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MATEMATIKA II Kolokvij 4.5.2016. γ

1. a) (6 bodova) Riješite diferencijalnu jednadžbu

$$y'' - 9y' + 8y = 6e^{2x}.$$

b) (7 bodova) Što je to linearna diferencijalna jednadžba 2. reda s konstantnim koeficijentima? Ima li takva jednadžba uvijek (jedinstveno) rješenje? Ako ima, kako ga računamo?

a) $\lambda^2 - 9\lambda + 8 = 0 \qquad \lambda_{1,2} = \dfrac{9 \pm \sqrt{81-32}}{2} = \dfrac{9 \pm 7}{2} = 1, 8$

$y_H = C_1 e^{x} + C_2 e^{8x}, \quad C_1, C_2 \in \mathbb{R}$

$$\left.\begin{aligned} y_P &= Ae^{2x} \\ y_P' &= 2Ae^{2x} \\ y_P'' &= 4Ae^{2x} \end{aligned}\right\} \quad \begin{aligned} 4Ae^{2x} - 18Ae^{2x} + 8Ae^{2x} &= 6e^{2x} \\ -6Ae^{2x} &= 6e^{2x} \\ -6A = 6 \Rightarrow A &= -1 \end{aligned}$$

$y_P = -e^{2x}$

$$\boxed{y(x) = C_1 e^{x} + C_2 e^{8x} - e^{2x}, \quad C_1, C_2 \in \mathbb{R}}$$

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1. a) (7 bodova) Riješite diferencijalnu jednadžbu:

$$y'' - 3y' + 2y = 4\sin x - 2\cos x.$$

b) (7 bodova) Što je to Cauchyev problem 1. reda? Uz koje uvjete dopušta rješenje, a uz koje uvjete dopušta jedinstveno rješenje?

a) $\lambda^2 - 3\lambda + 2 = 0$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = 1, 2$$

$$y_H = C_1 e^x + C_2 e^{2x}, \quad C_1, C_2 \in \mathbb{R}$$

$$y_p = A\sin x + B\cos x$$

$$y_p' = A\cos x - B\sin x$$

$$y_p'' = -A\sin x - B\cos x$$

$$-A\sin x - B\cos x - 3A\cos x + 3B\sin x + 2A\sin x + 2B\cos x = 4\sin x - 2\cos x$$

$$-A + 3B + 2A = 4 \Rightarrow A + 3B = 4 \Rightarrow A = 4 - 3B$$

$$-B - 3A + 2B = -2 \Rightarrow B - 3A = -2$$

$$B - 12 + 9B = -2$$

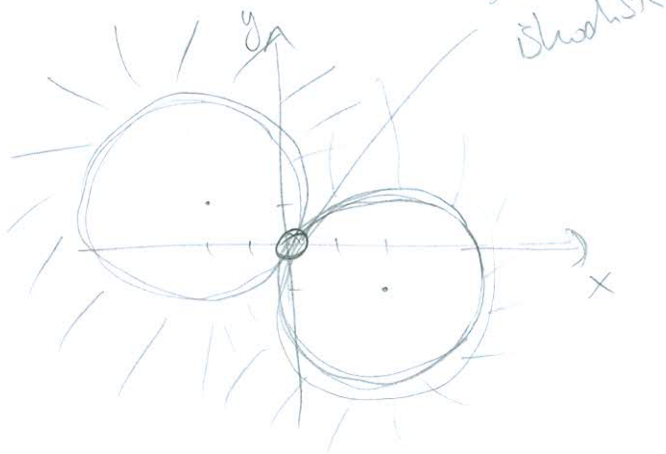
$$10B = 10 \quad \boxed{B = 1} \quad \boxed{A = 1}$$

$$y_p(x) = \sin x + \cos x \qquad \boxed{y(x) = C_1 e^x + C_2 e^{2x} + \sin x + \cos x \quad C_1, C_2 \in \mathbb{R}}$$

~~2.~~ a) (5 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = e^{\arccos\left(\frac{2y-4x}{x^2+y^2}\right)}.$$

b) (6 bodova) Nađite tangencijalnu ravninu na plohu $z = x^2 + y^2$ koja je okomita na pravac $p \ldots \frac{x-1}{2} = \frac{y+2}{-4} = \frac{z}{-1}$.



a) $x^2+y^2 \neq 0 \Rightarrow (x,y) \neq (0,0)$

$$-1 \leq \frac{2y-4x}{x^2+y^2} \leq 1 \quad \Big| \cdot (x^2+y^2)$$

$$-x^2-y^2 \leq 2y-4x \leq x^2+y^2$$

$-x^2+4x-y^2-2y \leq 0$

$x^2-4x+y^2+2y \geq 0$

$x^2-4x+4+y^2+2y+1 \geq 5$

$(x-2)^2+(y+1)^2 \geq 5$

$S(2,-1), \; r=\sqrt{5}$

$2y-4x \leq x^2+y^2$

$x^2+4x+y^2-2y \geq 0$

$x^2+4x+4+y^2-2y+1 \geq 5$

$(x+2)^2+(y-1)^2 \geq 5$

$S(-2,1), \; r=\sqrt{5}$

$$D = \{ (x,y) \in \mathbb{R}^2 : (x-2)^2+(y+1)^2 \geq 5, \; (x+2)^2+(y-1)^2 \geq 5, \; x^2+y^2 \neq 0 \}$$

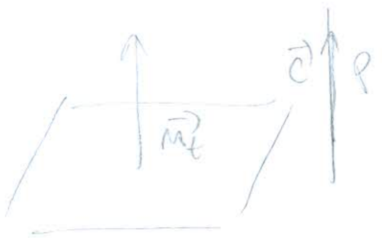
b) $\frac{\partial f}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2y$

$\Pi_t \ldots \quad 2x_0(x-x_0) + 2y_0(y-y_0) = z - z_0$

$$2x_0x - 2x_0^2 + 2y_0y - 2y_0^2 = z - x_0^2 - y_0^2$$

$$2x_0x + 2y_0y - z - x_0^2 - y_0^2 = 0$$



$\vec{n}_t = (2x_0, 2y_0, -1)$

$\vec{n}_t \parallel \vec{c} = (2, -4, -1)$

$$\frac{2x_0}{2} = \frac{2y_0}{-4} = \frac{-1}{-1} = 1$$

$2x_0 = 2 \Rightarrow \quad x_0 = 1$

$2y_0 = -4 \Rightarrow \quad y_0 = -2 \qquad T(1, -2, 5)$

$z_0 = x_0^2 + y_0^2 = 1 + 4 = 5$

$$\boxed{\Pi_t \ldots \quad 2x - 4y - z - 5 = 0}$$

3. a) (7 bodova) Ispitajte ekstreme funkcije:

$$f(x,y) = x^3 + x^2 + 8xy + 4y^2 - 8x - 8y + 3.$$

b) (6 bodova) Uz koje uvjete realna funkcija dvije realne varijable ima lokalni maksimum? Ilustrirajte skicom.

a) $D = \mathbb{R}^2$

$$\frac{\partial f}{\partial x} = 3x^2 + 2x + 8y - 8 = 0$$

$$\frac{\partial f}{\partial y} = 8x + 8y - 8 = 0 \Rightarrow x + y = 1$$

$$x = 1 - y$$

$$3(1-y)^2 + 2(1-y) + 8y = 8$$

$$3(1 - \cancel{2y} + y^2) + 2 \cancel{- 2y} \cancel{+ 8y} = 8$$

$$3y^2 + 5 = 8 \Rightarrow 3y^2 = 3 \Rightarrow y = \pm 1$$

$$x_1 = 0, \quad x_2 = 2$$

$$T_1(0,1) \quad T_2(2,-1)$$

$$\frac{\partial^2 f}{\partial x^2} = 6x + 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 8$$

$$\frac{\partial^2 f}{\partial y^2} = 8$$

$T_1 \Rightarrow A = 2, \ B = 8, \ C = 8$

$AC - B^2 = 16 - 64 < 0$

$\Rightarrow$ u T_1 nema ekstrema

$T_2 \Rightarrow A = 14, \ B = 8, \ C = 8$

$AC - B^2 = 14 \cdot 8 - 8^2 > 0$

$A > 0$ u T_2 lok. min.

4. a) (7 bodova) Skicirajte i odredite površinu manjem od dva lika omeđena krivuljama

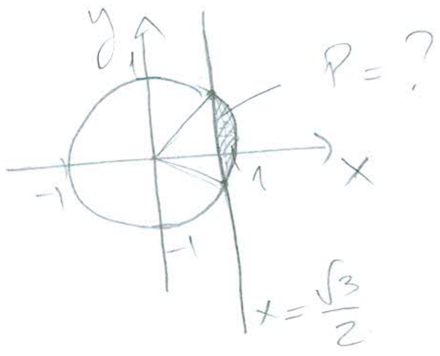
$$r = 1, \quad r\cos\varphi = \frac{\sqrt{3}}{2}.$$

b) (7 bodova) Što to znači da je realna funkcija dvije realne varijable integrabilna na zatvorenom i omeđenom području? Kako računamo dvostruki integral integrabilne funkcije na pravokutniku?

a) $r = 1$

$\underbrace{r\cos\varphi} = \frac{\sqrt{3}}{2}$

$x = \frac{\sqrt{3}}{2} \Rightarrow$ pravac!



$$\left.\begin{aligned} r\cos\varphi &= \frac{\sqrt{3}}{2} \\ r &= 1 \end{aligned}\right\} \quad \cos\varphi = \frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$$

$$-\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{6}$$

$$\frac{\sqrt{3}}{2\cos\varphi} \leq r \leq 1$$

$$P = \int_{-\pi/6}^{\pi/6} d\varphi \int_{\frac{\sqrt{3}}{2\cos\varphi}}^{1} r\,dr = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\varphi \left.\frac{r^2}{2}\right|_{\frac{\sqrt{3}}{2\cos\varphi}}^{1} = \frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(1 - \frac{3}{4\cos^2\varphi}\right) d\varphi$$

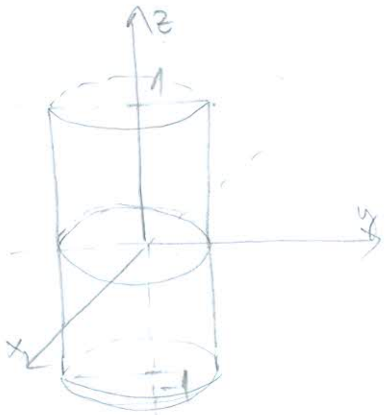
$$= \frac{1}{2}\left(\varphi\Big|_{-\pi/6}^{\pi/6} - \frac{3}{4}\operatorname{tg}\varphi\Big|_{-\pi/6}^{\pi/6}\right) = \frac{1}{2}\left(\frac{\pi}{6} + \frac{\pi}{6} - \frac{3}{4}\left(\operatorname{tg}\frac{\pi}{6} - \operatorname{tg}\frac{-\pi}{6}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{3} - \frac{\cancel{3}}{4}\cdot\left(\frac{\sqrt{3}}{\cancel{3}} + \frac{\sqrt{3}}{\cancel{3}}\right)\right)$$

5. (8 bodova) Izračunajte

$$\iiint_V e^{\sqrt{x^2+y^2}}\,dx\,dy\,dz,$$

gdje je V područje omeđenom plohama $x^2+y^2=4$, $z=-1$ i $z=1$. Skicirajte područje integracije.



$0 \le \varrho \le 2$

$0 \le \varphi \le 2\pi$

$-1 \le z \le 1$

$$I = \int_0^{2\pi} d\varphi \int_0^2 d\varrho \int_{-1}^{1} e^{\sqrt{\varrho^2}} \cdot \varrho \, dz = \int_0^{2\pi} d\varphi \int_0^2 e^{\varrho}\varrho \, d\varrho \, \underbrace{z \Big|_{-1}^{1}}_{2} =$$

$$= 2 \underbrace{\int_0^{2\pi} d\varphi}_{\varphi \Big|_0^{2\pi} = 2\pi} \underbrace{\int_0^2 \varrho e^{\varrho} d\varrho}_{} = 2 \cdot 2\pi \cdot \left(\varrho e^{\varrho} \Big|_0^2 - \int_0^2 e^{\varrho} d\varrho \right) =$$

$$\left\{ \begin{array}{ll} u = \varrho & du = d\varrho \\ dv = e^{\varrho} d\varrho & v = e^{\varrho} \end{array} \right\}$$

$$= 4\pi \left(2e^2 - e^{\varrho} \Big|_0^2 \right) = 4\pi \left(2e^2 - e^2 + 1 \right) = 4\pi \left(e^2 + 1 \right)$$

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1. a) (6 bodova) Riješite diferencijalnu jednadžbu:

$$y^3 y' = \frac{1}{x^2}$$

uz početni uvjet $y(1) = 1$.

b) (7 bodova) Što je to linearna diferencijalna jednadžba 2. reda? Uz koje uvjete ona dopušta jedinstveno rješenje? Kako izgleda, i zašto, rješenje te jednadžbe?

a) $$y^3 y' = \frac{1}{x^2}$$

$$y^3 \frac{dy}{dx} = \frac{1}{x^2} \quad / dx$$

$$y^3 dy = \frac{1}{x^2} dx \quad / \int$$

$$\int y^3 dy = \int x^{-2} dx$$

$$\frac{y^4}{4} = -\frac{1}{x} + C \Rightarrow y^4 = -\frac{4}{x} + C_1 \Rightarrow y = \sqrt[4]{-\frac{4}{x} + C_1}$$

$$y(1) = \sqrt[4]{C_1 - 4} = 1 \Rightarrow C_1 - 4 = 1 \Rightarrow C_1 = 5$$

$$\boxed{y(x) = \sqrt[4]{5 - \frac{4}{x}}}$$

2. (a) (4 boda) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \operatorname{arctg}\left(\frac{1}{\sqrt{9x^2-y^2-36}}\right).$$

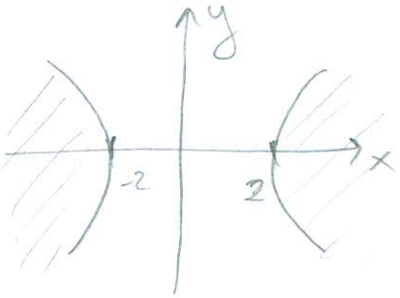
(b) (7 bodova) Nađite tangencijalne ravnine na plohu $z = 4 - x^2 - y^2$ koje prolaze točkom $O(0,0,5)$ i okomite su na ravninu $\pi \ldots x + y - 2z + 1 = 0$.

a) $\left.\begin{array}{l}\sqrt{9x^2-y^2-36} \neq 0 \\ 9x^2-y^2-36 \geq 0\end{array}\right\}$ $9x^2-y^2-36>0$

$9x^2-y^2>36 \Rightarrow x^2-\left(\frac{y}{3}\right)^2>4$

$\frac{x^2}{4}-\frac{y^2}{36}>1$

$\mathcal{D} = \{(x,y)\in\mathbb{R}^2 : 9x^2-y^2>36\}$



b) $\frac{\partial f}{\partial x} = -2x \qquad \frac{\partial f}{\partial y} = -2y$

$\pi_t \ldots -2x_0(x-x_0) - 2y_0(y-y_0) = z - z_0$

$-2x_0x + \cancel{2}x_0^2 - 2y_0y + \cancel{2}y_0^2 = z - 4 + \cancel{x_0^2} + y_0^2$

$-2x_0x - 2y_0y - z + x_0^2 + y_0^2 + 4 = 0$

$\vec{n}_t = (-2x_0, -2y_0, -1) \qquad \vec{n}_t \perp \vec{n} = (1,1,-2)$

$-2x_0\cdot 1 - 2y_0 \cdot 1 - 1\cdot(-2) = 0$

$-2x_0 - 2y_0 + 2 = 0 \Rightarrow x_0 + y_0 = 1 \Rightarrow x_0 = 1 - y_0$

$O(0,0,5) \in \pi_t \Rightarrow -5 + x_0^2 + y_0^2 + 4 = 0$

$1 - 2y_0 + 2y_0^2 - 1 = 0$

$y_0(y_0-1) = 0$

$y_0 = 0 \qquad y_0 = 1$

3. a) (7 bodova) Nađite točke na paraboli $y = x^2 + 1$ koje su najbliže točki $A(0.3)$.

b) (6 bodova) Uz koje uvjete realna funkcija dvije realne varijable ima lokalni minimum? Ilustrirajte skicom.

a) $f(x,y) = d(T,A) = \sqrt{(x-0)^2 + (y-3)^2} = \sqrt{x^2 + (y-3)^2}$

$L(x,y,\lambda) = x^2 + (y-3)^2 + \lambda \cdot \underbrace{(y - x^2 - 1)}_{g(x,y)}$

$\frac{\partial L}{\partial x} = 2x - 2\lambda x = 0 \Rightarrow x(1-\lambda) = 0$

$\frac{\partial L}{\partial y} = 2(y-3) + \lambda = 0$

$x = 0$ $\quad$ $1 - \lambda = 0$

$\lambda = 1$

$y = x^2 + 1$

$y = 1$

$2y - 6 + 1 = 0$

$2y = 5$

$y = \frac{5}{2}$

$x^2 = y - 1 = \frac{5}{2} - 1 = \frac{3}{2}$

$x = \pm\sqrt{\frac{3}{2}}$

$T_1(0,1)$ $\quad$ $T_2\left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$

$T_3\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$

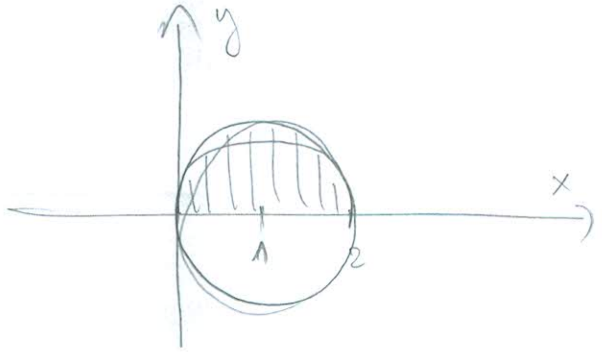
$d(T_2, A) = d(T_3, A) = \sqrt{\frac{3}{2} + \frac{1}{4}} = \sqrt{\frac{7}{4}} \Rightarrow T_2, T_3$ su najbliže

$d(T_1, A) = \sqrt{4} = 2$

4. a) (7 bodova) Izračunajte integral

$$\iint_D \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy,$$

gdje je $D = \{(x,y) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 2,\ y \geqslant 0,\ x^2+y^2 \leqslant 2x\}$. Skicirajte područje integracije.



$$x^2+y^2 \le 2x$$
$$x^2-2x+1+y^2 \le 1$$
$$(x-1)^2+y^2 \le 1$$

$$r^2 \le 2\cancel{r}\cos\varphi$$
$$r \le 2\cos\varphi$$

$$0 \le \varphi \le \frac{\pi}{2}$$
$$0 \le r \le 2\cos\varphi$$

$$I = \int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} \cancel{r}\cdot\frac{1}{\cancel{\sqrt{r^2}}}\,dr = \int_0^{\pi/2} d\varphi\ r\Big|_0^{2\cos\varphi}$$

$$= \int_0^{\pi/2} 2\cos\varphi\,d\varphi = 2\sin\varphi\Big|_0^{\pi/2} = 2\sin\frac{\pi}{2}$$
$$= 2$$

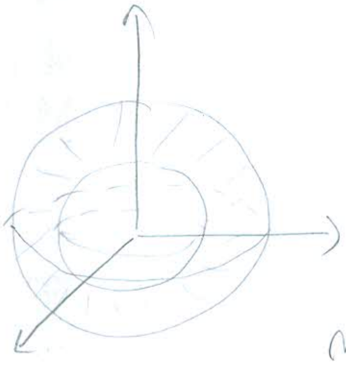
5. a) (9 bodova) Izračunajte masu tijela omeđenog plohama $x^2+y^2+z^2=1$ i $x^2+y^2+z^2=9$, ako mu je gustoća dana s

$$\rho(x,y,z)=(x^2+y^2)^2.$$

Skicirajte tijelo.

b) (7 bodova) Što to znači da je realna funkcija tri realne varijable integrabilna na zatvorenom i omeđenom području? Kako računamo trostruki integral integrabilne funkcije na kvadru?

a)



$0 \le \varphi \le 2\pi$

$0 \le \theta \le \pi$

$1 \le r \le 3$

$$m = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_1^3 r^2 \sin\theta \cdot (r^2 \sin^2\theta)^2 \, dr$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^5\theta \, d\theta \underbrace{\int_1^3 r^6 \, dr}_{\frac{r^7}{7}\Big|_1^3 = \frac{3^7-1^7}{7}} = \frac{3^7-1^7}{7} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^5\theta \, d\theta$$

$$= \frac{3^7-1}{7} \underbrace{\int_0^{2\pi} d\varphi}_{\varphi\Big|_0^{2\pi}} \int_0^{\pi} \underbrace{\sin^4\theta}_{(\sin^2\theta)^2} \cdot \sin\theta \, d\theta = \frac{3^7-1}{7} \cdot 2\pi \int_0^{\pi} (1-\cos^2\theta)^2 \sin\theta$$

$$\left\{ \begin{array}{l} t = \cos\theta \\ dt = -\sin\theta \, d\theta \\ 0 \to 1 \\ \pi \to -1 \end{array} \right\}$$

$$= \frac{3^7-1}{7} \cdot 2\pi \cdot \left[-\int_1^{-1} (1-t^2)^2 \, dt \right]$$

$$= 2\pi \, \frac{3^7-1}{7} \int_{-1}^{1} (1-2t^2+t^4) \, dt = 2\pi \, \frac{3^7-1}{7} \left(t - \frac{2}{3}t^3 + \frac{t^5}{5} \right) \Big|_{-1}^{1}$$

$$= 2\pi \, \frac{3^7-1}{7} \cdot \frac{16}{15} = \boxed{\frac{32}{105}(3^7-1)\pi}$$

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1. a) (7 bodova) Riješite diferencijalnu jednadžbu
$$y' = \sin x - y^2 \sin x.$$
b) (7 bodova) Što je to Cauchyev problem 2. reda? Uz koje uvjete dopušta rješenje, a uz koje uvjete dopušta jedinstveno rješenje?

a) $y' = \sin x - y^2 \sin x$

$y' = \sin x (1-y^2)$

$\frac{dy}{dx} = \sin x (1-y^2) \quad / \frac{dx}{1-y^2}$

$\frac{dy}{1-y^2} = \sin x \, dx \quad / \int$

$\int \frac{dy}{1-y^2} = \int \sin x \, dx$

$\int \frac{\frac{1}{2}}{1-y} dy + \int \frac{\frac{1}{2}}{1+y} dy = -\cos x + C$

$-\frac{1}{2} \ln|1-y| + \frac{1}{2} \ln|1+y| = -\cos x + C$

$\ln\sqrt{\frac{1+y}{1-y}} = -\cos x + C \quad / e^{\wedge}$

$\sqrt{\frac{1+y}{1-y}} = C_1 e^{-\cos x} \quad /^2$

$\frac{1+y}{1-y} = C_1 e^{-2\cos x}$

$\frac{1}{1-y^2} = \frac{1}{(1-y)(1+y)}$

$\frac{1}{1-y^2} = \frac{A}{1-y} + \frac{B}{1+y} \quad / \cdot 1-y^2$

$1 = A(1+y) + B(1-y)$

$1 = A + Ay + B - By$

$A + B = 1$

$A - B = 0 \Rightarrow A = B$

$A = \frac{1}{2}, B = \frac{1}{2}$

$\boxed{y(x) = \frac{1 - C_1 e^{2\cos x}}{1 + C_1 e^{2\cos x}}}$

2. a) (6 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x,y) = \sqrt[5]{\ln\left(\frac{x}{y}\right) + \arcsin\frac{x}{x^2+y^2}}.$$

b) (6 bodova) Nađite tangencijalne ravnine na plohu $x^2 + 2y^2 + z^2 = 13$ koje su paralelne s ravninom $\pi \ldots x + 4y + 2z - 1 = 0$.

a) $\frac{x}{y} > 0$

$-1 \leq \frac{x}{x^2+y^2} \leq 1 \quad / \; x^2+y^2$

$-x^2-y^2 \leq x \leq x^2+y^2$

$-x^2 - x - y^2 \leq 0$ $\qquad$ $x - x^2 - y^2 \leq 0$

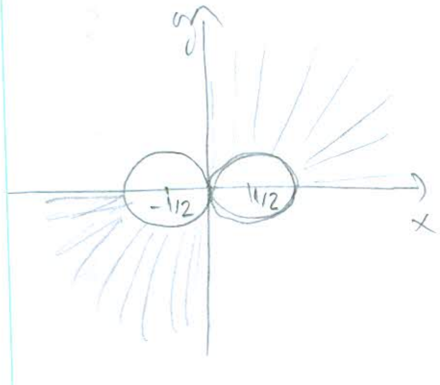
$x^2 + x + y^2 \geq 0$ $\qquad$ $x^2 - x + y^2 \geq 0$

$\underbrace{x^2 + x + \frac{1}{4}} + y^2 \geq \frac{1}{4}$ $\qquad$ $\underbrace{x^2 - x + \frac{1}{4}} + y^2 \geq \frac{1}{4}$

$\left(x + \frac{1}{2}\right)^2 + y^2 \geq \frac{1}{4}$ $\qquad$ $\left(x - \frac{1}{2}\right)^2 + y^2 \geq \frac{1}{4}$

$S\left(-\frac{1}{2}, 0\right) \quad r = \frac{1}{2}$ $\qquad$ $S\left(\frac{1}{2}, 0\right) \quad r = \frac{1}{2}$

$D = \left\{(x,y) \in \mathbb{R}^2 : \frac{x}{y} > 0, \; \left(x + \frac{1}{2}\right)^2 + y^2 \geq \frac{1}{4}, \; \left(x - \frac{1}{2}\right)^2 + y^2 \geq \frac{1}{4} \right)$



b) $\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 4y \qquad \frac{\partial f}{\partial z} = 2z$

$$\Pi_t \ldots \; 2x_0(x-x_0) + 4y_0(y-y_0) + 2z_0(z-z_0) = 0$$

$$2x_0x - 2x_0^2 + 4y_0y - 4y_0^2 + 2z_0z - 2z_0^2 = 0 \quad |:2$$

$$x_0x + 2y_0y + z_0z - x_0^2 - 2y_0^2 - z_0^2 = 0$$

$$\vec{n}_t = (x_0, 2y_0, z_0) \parallel \vec{n} = (1,4,2)$$

$$\frac{x_0}{1} = \frac{\cancel{2}y_0}{\cancel{4}\,2} = \frac{z_0}{2} = t \qquad x_0 = t, \; y_0 = 2t, \; z_0 = 2t$$

$$x_0^2 + 2y_0^2 + z_0^2 = 13$$

$$t^2 + 2\cdot 4t^2 + 4t^2 = 13$$

$$13t^2 = 13 \qquad t = \pm 1 \Rightarrow \quad T_1(1,2,2), \; T_2(-1,-2,-2)$$

$$\Pi_{t_1} \ldots \; \boxed{x + 4y + 2z - 13 = 0}$$

$$\Pi_{t_2} \ldots \; -x - 4y - 2z - 13 = 0$$

$$\boxed{x + 4y + 2z + 13 = 0}$$

3. a) (5 bodova) Odredite lokalni ekstrem funkcije

$$f(x,y) = x^2 + (y-1)^2 + x(y-1).$$

b) (6 bodova) Što je to diferencijal realne funkcije dvije realne varijable? Koja je njegova geometrijska interpretacija? Ilustrirajte skicom.

a) $\frac{\partial f}{\partial x} = 2x + y - 1 \Rightarrow y = -2x + 1$

$\frac{\partial f}{\partial y} = 2(y-1) + x = 0 \Rightarrow 2(-2x+1-1) + x = 0$

$-4x + x = 0$

$-3x = 0$

$\boxed{x = 0 \quad y = 1}$

$\boxed{T(0,1)}$

$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 1 \qquad \frac{\partial^2 f}{\partial y^2} = 2$

$\overset{\shortparallel}{A} \qquad \overset{\shortparallel}{B} \qquad \overset{\shortparallel}{C} \qquad AC - B^2 = 4 - 1 = 3 > 0$

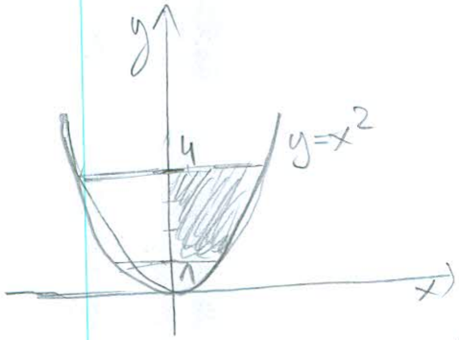
> 0

$\Rightarrow$ u T je lok. min

4. a) (7 bodova) Izračunajte integral

$$\iint_D e^{\frac{x}{\sqrt{y}}}\,dx\,dy,$$

gdje je $D = \{(x,y) \in \mathbb{R}^2 \colon 1 \le y \le 4,\ 0 \le x \le \sqrt{y}\}$. Skicirajte D.



$x \le \sqrt{y}$

$x^2 \le y$

$$I = \int_1^4 dy \int_0^{\sqrt{y}} e^{\frac{x}{\sqrt{y}}}\,dx = \int_1^4 dy\ \sqrt{y}\, e^{\frac{x}{\sqrt{y}}}\Big|_0^{\sqrt{y}} =$$

$$= \int_1^4 \sqrt{y}\,(e-1)\,dy = (e-1)\,\frac{y^{3/2}}{3/2}\Big|_1^4 = \frac{2}{3}(e-1)\sqrt{y^3}\Big|_1^4$$

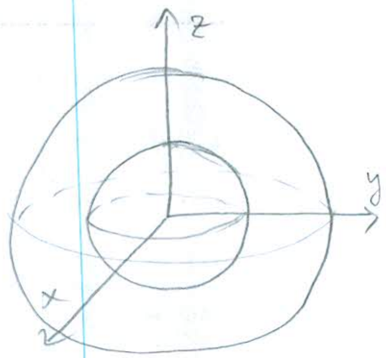
$$= \frac{2}{3}(e-1)\left(\sqrt{64}-1\right) = \frac{14}{3}(e-1)$$

5. a) (9 bodova) Izračunajte moment inercije tijela gustoće

$$\rho(x,y,z) = x^2 + y^2,$$

omeđenog plohama $x^2+y^2+z^2=1$ i $x^2+y^2+z^2=4$, s obzirom na os z. Skicirajte tijelo.

b) (7 bodova) Opišite postupak zamjene varijabli u trostrukom integralu.



$$0 \le \varphi \le 2\pi$$
$$0 \le \theta \le \pi$$
$$1 \le r \le 2$$

$$I_z = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_1^2 \overbrace{r^2 \sin^2\theta}^{x^2+y^2} \cdot \overbrace{r^2 \sin^2\theta}^{\rho(x,y,z)} \cdot r^2 \sin\theta \, dr =$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_1^2 r^6 \sin^5\theta \, dr = \underbrace{\int_0^{2\pi} d\varphi}_{\varphi \Big|_0^{2\pi} = 2\pi} \int_0^{\pi} \sin^5\theta \, d\theta \underbrace{\frac{r^7}{7}\Big|_1^2}_{\frac{2^7-1}{7}}$$

$$= 2\pi \cdot \frac{2^7-1}{7} \cdot \int_0^{\pi} \sin^4\theta \sin\theta \, d\theta =$$

$$= 2\pi \cdot \frac{2^7-1}{7} \cdot \int_0^{\pi} (1-\cos^2\theta)^2 \sin\theta \, d\theta = \left\{ \begin{array}{ll} t = \cos\theta & 0 \to 1 \\ dt = -\sin\theta \, d\theta & \pi \to -1 \end{array} \right\}$$

$$= -2\pi \, \frac{2^7-1}{7} \int_1^{-1} (1 - 2t^2 + t^4) \, dt =$$

$$= \frac{2\pi}{7} \cdot (2^7-1) \underbrace{\left(t - \frac{2}{3}t^3 + \frac{t^5}{5}\right)\Big|_{-1}^{1}}_{2 - \frac{4}{3} + \frac{2}{5} = \frac{16}{15}} = \frac{32\pi}{15 \cdot 7} \overbrace{(2^7-1)}^{127} = \frac{127 \cdot 32}{15 \cdot 7}\pi$$

$$= \boxed{\frac{4064}{105}\pi}$$

~~2.~~ a) (6 bodova) Odredite i skicirajte prirodnu domenu funkcije

$$f(x, y) = \frac{1}{\ln(x - y^2 + 1)}.$$

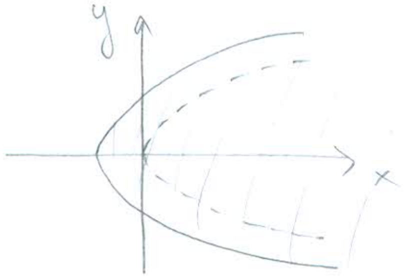
b) (6 bodova) Nađite tangencijalne ravnine na plohu $z = xy$ koje su paralelne s ravninom $\pi \ldots 2x + y + z - 3 = 0$.

a) $x - y^2 + 1 > 0 \Rightarrow x > y^2 - 1$

$\ln(x - y^2 + 1) \neq 0 \Rightarrow x - y^2 + 1 \neq 1$

$x - y^2 \neq 0 \Rightarrow x \neq y^2$

$\mathcal{D} = \{ (x,y) \in \mathbb{R}^2 : x > y^2 - 1 , x \neq y^2 \}$



b) $\frac{\partial f}{\partial x} = y \quad \frac{\partial f}{\partial y} = x$

$\Pi_t \ldots \quad y_0(x - x_0) + x_0(y - y_0) = z - z_0$

$y_0 x - x_0 y_0 + x_0 y - \cancel{x_0 y_0} = z - \cancel{x_0 y_0}$

$y_0 x + x_0 y - z - x_0 y_0 = 0$

$\vec{n}_t = (y_0, x_0, -1) \qquad \vec{n}_t \parallel \vec{n} = (2, 1, 1)$

$\frac{y_0}{2} = \frac{x_0}{1} = \frac{-1}{1} = -1 \Rightarrow \boxed{\begin{matrix} y_0 = -2 \\ x_0 = -1 \end{matrix}}$

$\Pi_t \ldots \quad -2x - y - z - 2 = 0$

$\boxed{2x + y + z + 2 = 0}$

3. a) (7 bodova) Ispitajte ekstreme funkcije:

$$f(x, y) = \frac{2}{y} + \frac{y^2}{x} + x.$$

b) (6 bodova) Što su to parcijalne derivacije realne funkcije dvije realne varijable? Koja je njihova geometrijska interpretacija? Ilustrirajte skicom.

a) $x, y \neq 0$ $\quad \frac{\partial f}{\partial x} = -\frac{y^2}{x^2} + 1 = 0 \,/ x^2 \quad \frac{\partial f}{\partial y} = \frac{-2}{y^2} + \frac{2y}{x} = 0 \,/ \cdot x y^2$

$-y^2 + x^2 = 0 \qquad -2x + 2y^3 = 0$

$y^2 = x^2 \qquad x - y^3 = 0$

$y^2 = (y^3)^2 \qquad x = y^3$

$y^2 = y^6$

$y^2(1 - y^4) = 0$

$\cancel{y = 0} \quad y^4 = 1 \qquad T_1(1,1)$

$\boxed{y = \pm 1} \Rightarrow \boxed{x = \pm 1} \qquad T_2(-1,-1)$

$\frac{\partial^2 f}{\partial x^2} = \frac{2y^2}{x^3} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{-2y}{x^2} \qquad \frac{\partial^2 f}{\partial y^2} = \frac{4}{y^3} + \frac{2}{x}$

$T_1(1,1) \Rightarrow A = 2 \quad B = -2 \quad C = 6, \quad AC - B^2 = 12 - 4 > 0$

$> 0 \Rightarrow$ lok min u $T_1(1,1)$

$T_2(-1,-1) \Rightarrow A = -2 \quad B = 2 \quad C = -6, \quad AC - B^2 = 12 - 4 > 0$

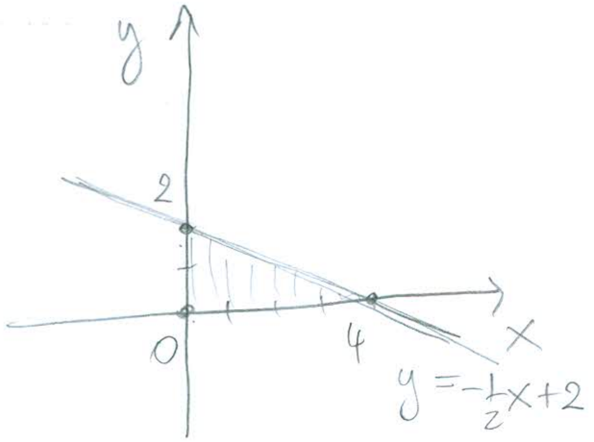
$< 0 \Rightarrow$ lok maks u $T_2(-1,-1)$

4. a) (8 bodova) Izračunajte masu trokuta s vrhovima $(0,0)$, $(0,2)$ i $(4,0)$, ako mu je plošna gustoća dana s

$$\rho(x,y) = xe^{2y}.$$

Skicirajte trokut.

b) (7 bodova) Opišite postupak zamjene varijabli u dvostrukom integralu.



$$0 \le x \le 4$$
$$0 \le y \le -\tfrac{1}{2}x + 2$$

$$\mu = \iint \rho(x,y)\,dx\,dy$$

$$\mu = \int_0^4 dx \int_0^{-\frac{1}{2}x+2} x e^{2y}\,dy = \int_0^4 x\,dx\ \frac{e^{2y}}{2}\Big|_0^{-\frac{1}{2}x+2}$$

$$= \frac{1}{2}\int_0^4 x\,dx\,(e^{-x+4} - e^0)$$

$$= \frac{1}{2}\Big[\int_0^4 e^{-x}e^4 x\,dx - \underbrace{\int_0^4 dx}_{x\big|_0^4 = 4}\Big]$$

$$= \left\{\begin{matrix} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x}\end{matrix}\right\}$$

$$= \frac{e^4}{2}\Big(-xe^{-x}\Big|_0^4 + \int_0^4 e^{-x}dx\Big) - \frac{1}{2}\cdot 4$$

$$= \frac{e^4}{2}\Big(-4e^{-4} - \underbrace{e^{-x}\Big|_0^4}_{e^{-4}-1}\Big) - 2 = \frac{e^4}{2}(-5e^{-4}+1) - 2$$

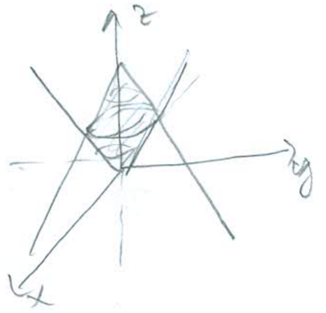
$$= -\frac{5}{2} + \frac{1}{2}e^4 - 2$$

$$= \frac{1}{2}e^4 - \frac{9}{2}$$

5. (7 bodova) Izračunajte

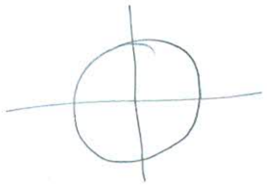
$$\iiint_V y^2\,dx\,dy\,dz,$$

gdje je V područje omeđeno plohama $z = 2 - \sqrt{x^2+y^2}$ i $z = \sqrt{x^2+y^2}$. Skicirajte područje integracije.



$$2-\sqrt{x^2+y^2} = \sqrt{x^2+y^2}$$

$$2-\rho = \rho \Rightarrow \rho = 1$$



$$0 \le \varphi \le 2\pi$$

$$0 \le \rho \le 1$$

$$\rho \le z \le 2-\rho$$

$$I = \int_0^{2\pi} d\varphi \int_0^1 d\rho \int_\rho^{2-\rho} \rho\,\rho^2 \sin^2\varphi\,dz = \int_0^{2\pi} \sin^2\varphi\,d\varphi \int_0^1 \rho^3 d\rho\; z\Big|_\rho^{2-\rho} =$$

$$= \int_0^{2\pi} \sin^2\varphi\,d\varphi \int_0^1 \underbrace{\rho^3(2-2\rho)}_{2\rho^3-2\rho^4}d\rho = \int_0^{2\pi} \sin^2\varphi\,d\varphi \left(\cancel{2}\frac{\rho^4}{\cancel{4}\,2} - \frac{2\rho^5}{5}\right)\Big|_0^1 =$$

$$= \int_0^{2\pi} \sin^2\varphi\,d\varphi \underbrace{\left(\frac{1}{2}-\frac{2}{5}\right)}_{\frac{1}{10}} = \frac{1}{10}\int_0^{2\pi} \frac{1-\cos 2\varphi}{2}\,d\varphi =$$

$$= \frac{1}{20}\left(\underbrace{\varphi\Big|_0^{2\pi}}_{=2\pi} - \underbrace{\frac{1}{2}\sin 2\varphi\Big|_0^{2\pi}}_{=0}\right) = \frac{2\pi}{20} = \frac{\pi}{10}$$