Poglavlje 3
Analitička geometrija

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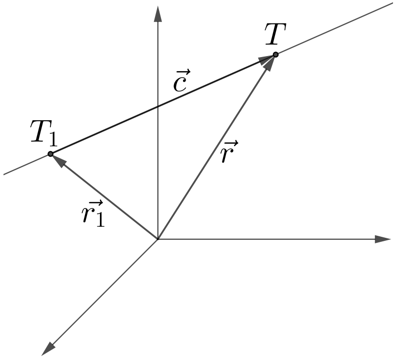
3.1 Pravac

Pravac je određen nekom točkom T_1 i vektorom smjera $\vec{c}$ u oznaci $p = (T_1, \vec{c})$.

Neka je $\vec{r}$ radijvektor neke proizvoljne točke T pravca p, a $\vec{r_1}$ radijvektor fiksne točke T_1 na pravcu p.

Vektorska jednadžba pravca je: $\vec{r} = \vec{r_1} + t \cdot \vec{c}$.

Ako je $T_1 = (x_1, y_1, z_1)$, $T = (x, y, z)$ i $\vec{c} = (l, m, n)$, vektorska jednadžba je:



$$x\vec{i}+y\vec{j}+z\vec{k} = x_1\vec{i}+y_1\vec{j}+z_1\vec{k}+tl\vec{i}+tm\vec{j}+tn\vec{k}$$

Izjednačavanjem koeficijenata dolazimo do parametarskog oblika jednadžbe pravca:

$$p \equiv \begin{cases} x = x_1 + tl \\ y = y_1 + tm \\ z = z_1 + tn \end{cases}$$

Eliminacijom parametra t dobivamo:

$$p \equiv \begin{cases} x = x_1 + tl \\ y = y_1 + tm \\ z = z_1 + tn \end{cases} \Longrightarrow \begin{cases} t = \dfrac{x - x_1}{l} \\ t = \dfrac{y - y_1}{m} \\ t = \dfrac{z - z_1}{n} \end{cases},$$

odnosno kanonski oblik jednadžbe pravca:

$$p \equiv \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n},$$

gdje su l, m i n skalarne komponente vektora smjera, tj. $\vec{c} = (l, m, n)$, a $T_1 = (x_1, y_1, z_1)$ je neka točka pravca p.

Ako je pravac zadan s dvije točke $T_1(x_1, y_1, z_1)$ i $T_2(x_2, y_2, z_2)$, onda je vektor smjera tog pravca

$$\vec{c} = \overrightarrow{T_1 T_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

Jednadžba pravca koji prolazi točkama T_1 i T_2 je:

$$p \equiv \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Zadatak (2.1.)

Parametarske jednadžbe pravca p zapiši u kanonskom obliku i odredi vektor smjera $\vec{c}$ ako je: a) $p \equiv \begin{cases} x = 1 + 2t \\ y = -2 + t \\ z = 3 - 2t \end{cases}$ b) $p \equiv \begin{cases} x = 2t \\ y = 1 \\ z = 1 + t \end{cases}$

Rješenje: a)

$$p \equiv \begin{cases} x = 1 + 2t \\ y = -2 + t \\ z = 3 - 2t \end{cases} \Longrightarrow \begin{cases} t = \dfrac{x-1}{2} \\ t = y + 2 \\ t = \dfrac{z-3}{-2} \end{cases}$$

$$\Longrightarrow p \equiv \frac{x-1}{2} = \frac{y+2}{1} = \frac{z-3}{-2}$$

Skalarne komponente vektora smjera vidimo iz nazivnika:
$\vec{c} = 2\vec{i} + \vec{j} - 2\vec{k}$

b) $p \equiv \begin{cases} x = 2t \\ y = 1 \\ z = 1 + t \end{cases} \implies \begin{cases} t = \dfrac{x}{2} \\ t = z - 1 \end{cases} \implies p \equiv \underbrace{\frac{x}{2} = \frac{y-1}{0} = \frac{z-1}{1}}_{\text{formalni zapis}}$

Skalarne komponente vektora smjera vidimo iz nazivnika: $\vec{c} = 2\vec{i} + \vec{k}$

Zadatak (2.2.)

Odredite kanonsku i parametarsku jednadžbu pravca koji prolazi

a) točkom $M(1,2,-1)$ i ima vektor smjera $\vec{c}=(1,3,-1)$

b) točkama $M(1,2,-1)$ i $N(2,0,3)$

c) točkama $M(1,2,-1)$ i $N(1,1,2)$ (sami)

Rješenje:

a) $p \equiv \begin{cases} x = 1 + t \\ y = 2 + 3t \\ z = -1 - t \end{cases}$ i $p \equiv \dfrac{x-1}{1} = \dfrac{y-2}{3} = \dfrac{z+1}{-1}$

b) Vektor smjera $\vec{c} = \overrightarrow{MN} \Longrightarrow \vec{c} = \vec{i} - 2\vec{j} + 4\vec{k}$

$$p \equiv \begin{cases} x = 1 + t \\ y = 2 - 2t \\ z = -1 + 4t \end{cases} \quad \text{i} \quad p \equiv \frac{x-1}{1} = \frac{y-2}{-2} = \frac{z+1}{4}$$

c) Vektor smjera $\vec{c} = \overrightarrow{MN} \Longrightarrow \vec{c} = -\vec{j} + 3\vec{k}$

$$p \equiv \begin{cases} x = 1 \\ y = 2 - t \\ z = -1 + 3t \end{cases} \quad \text{i} \quad p \equiv \frac{x-1}{0} = \frac{y-2}{-1} = \frac{z+1}{3}$$

Zadatak (2.3.)

Odredite nepoznati parametar λ tako da pravci $p_1 \equiv \dfrac{x+1}{1} = \dfrac{y}{-3} = \dfrac{z-1}{4}$ i $p_2 \equiv \dfrac{x-3}{\lambda} = \dfrac{y+2}{4} = \dfrac{z-4}{-5}$ imaju presječnu točku i odredite je.

Rješenje:

Zapišimo jednadžbe pravaca u parametarskom obliku. Za različite pravce imamo različite parametre.

$$p_1 \equiv \begin{cases} x = -1 + t \\ y = -3t \\ z = 1 + 4t \end{cases} \quad \text{i} \quad p_2 \equiv \begin{cases} x = 3 + \lambda s \\ y = -2 + 4s \\ z = 4 - 5s \end{cases}$$

$$p_1 \cap p_2 \neq \emptyset \Longrightarrow \begin{cases} -1 + t = 3 + \lambda s \\ \quad -3t = -2 + 4s \\ 1 + 4t = 4 - 5s \end{cases}$$

$$\begin{array}{rcl} -1 + t & = & 3 + \lambda s \\ -3t & = & -2 + 4s \quad / \cdot 4 \\ 1 + 4t & = & 4 - 5s \quad / \cdot 3 \\ \hline -12t & = & -8 + 16s \\ 3 + 12t & = & 12 - 15s \\ \hline 3 & = & 4 + s \end{array}$$

$$\Longrightarrow s = -1$$

$$-3t = -6 \Longrightarrow t = 2$$

$$\begin{aligned} -1+t &= 3+\lambda s \\ -1+2 &= 3-\lambda \\ \lambda &= 2 \end{aligned}$$

$$\left.\begin{aligned} x &= -1+t \\ y &= -3t \\ z &= 1+4t \end{aligned}\right\} \Longrightarrow \begin{aligned} x &= 1 \\ y &= -6 \\ z &= 9 \end{aligned}$$

Točka presjeka je $S = (1, -6, 9)$

Kut među pravcima $p_1 = (T_1, \vec{c_1})$ i $p_2 = (T_2, \vec{c_2})$ je kut među njihovim vektorima smjera, tj. $\varphi = \angle(\vec{c_1}, \vec{c_2}), \varphi \in \left[0, \frac{\pi}{2}\right]$

Neka je $\vec{c_1} = (l_1, m_1, n_1)$ i $\vec{c_2} = (l_2, m_2, n_2)$. Tada je

$$\cos\varphi = \frac{\vec{c_1} \cdot \vec{c_2}}{|\vec{c_1}| \cdot |\vec{c_2}|} = \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}}$$

Zadatak (2.4.)

Odredite kut među pravcima:

a) $p_1 \equiv \dfrac{x-2}{-3} = \dfrac{y+4}{-4} = \dfrac{z-4}{0}$ i $p_2 \equiv \dfrac{x+1}{-3} = \dfrac{y-2}{-4} = \dfrac{z+3}{5}$. ;

b) $p_1 \equiv \dfrac{x-1}{1} = \dfrac{y}{-2} = \dfrac{z-4}{-7}$ i $p_2 \equiv \dfrac{x+6}{5} = \dfrac{y-2}{1} = \dfrac{z-3}{1}$.

Rješenje:

a) $\vec{c_1} = (-3, -4, 0)$, $\vec{c_2} = (-3, -4, -5)$, $\varphi = \angle(p_1, p_2) = \angle(\vec{c_1}, \vec{c_2})$

$$\begin{aligned}
\cos\varphi &= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}} \\
&= \frac{-3 \cdot (-3) - 4 \cdot (-4) + 0 \cdot 5}{\sqrt{(-3)^2 + (-4)^2 + 0^2} \cdot \sqrt{(-3)^2 + (-4)^2 + 5^2}} \\
&= \frac{25}{\sqrt{25} \cdot \sqrt{50}} \\
&= \frac{25}{25\sqrt{2}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

$\varphi = 45^\circ$

b) $\vec{c_1} = (1, -2, -7)$, $\vec{c_2} = (5, 1, 1)$, $\varphi = \angle(p_1, p_2) = \angle(\vec{c_1}, \vec{c_2})$

$$\begin{aligned}
\cos\varphi &= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \cdot \sqrt{l_2^2 + m_2^2 + n_2^2}} \\
&= \frac{1 \cdot 5 - 2 \cdot 1 - 7 \cdot 1}{\sqrt{1^2 + (-2)^2 + (-7)^2} \cdot \sqrt{5^2 + 1^2 + 1^2}} \\
&= \frac{-4}{\sqrt{54} \cdot \sqrt{27}} \\
&= \frac{-4}{27\sqrt{2}}
\end{aligned}$$

$\varphi = 96^\circ 1'$, pa kako je $\varphi > 90^\circ$ zamjenjujemo ga s njegovim suplementarnim kutem.

$\angle(p_1, p_2) = 180^\circ - 96^\circ 1' = 83^\circ 59'$

Zadatak (2.5.)

Pravac p prolazi točkama $A(-2,1,3)$ i $B(0,-1,2)$. Odredite kuteve što ga zatvara s koordinatnim osima.

Rješenje:

$$\vec{c} = \overrightarrow{AB} = 2\vec{i} - 2\vec{j} - \vec{k}$$

$$|\vec{c}| = \sqrt{2^2 + 2^2 + 1^2} = 3$$

$$\alpha = \angle(\vec{c}, \vec{i}) \qquad \cos\alpha = \frac{x_{\vec{c}}}{|\vec{c}|} = \frac{2}{3} \qquad \alpha = 48^\circ 11'$$

$$\beta = \angle(\vec{c}, \vec{j}) \qquad \cos\beta = \frac{y_{\vec{c}}}{|\vec{c}|} = -\frac{2}{3} \qquad \beta = 131^\circ 48'$$

$$\gamma = \angle(\vec{c}, \vec{k}) \qquad \cos\gamma = \frac{z_{\vec{c}}}{|\vec{c}|} = -\frac{1}{3} \qquad \gamma = 109^\circ 28'$$

$$\beta > 90^\circ \Rightarrow \begin{array}{l} \beta = 180^\circ - 131^\circ 48' \\ \beta = 48^\circ 12' \end{array} \qquad \gamma > 90^\circ \Rightarrow \begin{array}{l} \gamma = 180^\circ - 109^\circ 28' \\ \gamma = 70^\circ 32' \end{array}$$

Zadatak (2.6.)

Zadan je pravac $p \equiv \frac{x-3}{2} = \frac{y+1}{1} = \frac{z-1}{4}$.

a) Leži li točka $A(1,0,-3)$ na pravcu p?

b) Prolazi li pravac p ishodištem $O(0,0,0)$?

c) Ako ne prolazi, odredite jednadžbu pravca p_1 za kojeg vrijedi: $p||p_1$ i $O(0,0,0) \in p_1$.

Rješenje: a)

$$A(1,0,-3) \in^? p$$

$$\frac{x-3}{2} = \frac{y+1}{1} = \frac{z-1}{4}$$

$$\frac{1-3}{2} = \frac{1}{1} = \frac{-3-1}{4}$$

$$-1 \neq 1 \neq -1$$

$$A \notin p$$

b)

$$O(0,0,0) \in^{?} p$$

$$\frac{x-3}{2} = \frac{y+1}{1} = \frac{z-1}{4}$$

$$\frac{0-3}{2} = \frac{0}{1} = \frac{0-1}{4}$$

$$-\frac{3}{2} \neq 0 \neq -\frac{-1}{4}$$

$$O \notin p$$

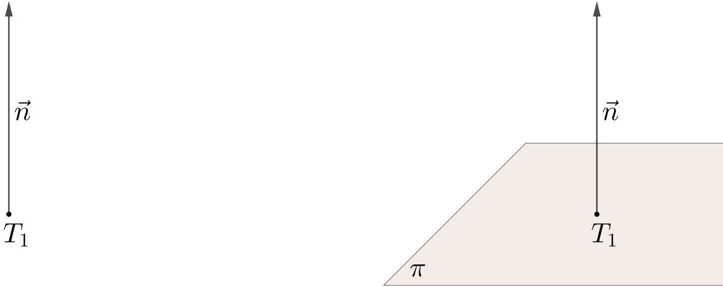
c)

$$\vec{c_1} = \vec{c} = 2\vec{i} + \vec{j} + 4\vec{k}$$

$$O \in p_1 \Longrightarrow p_1 \equiv \frac{x-0}{2} = \frac{y-0}{1} = \frac{z-0}{4}, \text{ tj. } p_1 \equiv \frac{x}{2} = y = \frac{z}{4}$$

3.2 Ravnina

Ravnina π određena je jednom svojom točkom $T_1 = (x_1, y_1, z_1)$ i vektorom normale $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$. Tako definiranu ravninu označavamo s: $\pi = (T_1, \vec{n})$.



Neka je $T=(x,y,z)$ proizvoljna točka ravnine π. Tada je

$$\overrightarrow{T_1T}=(x-x_1)\vec{i}+(y-y_1)\vec{j}+(z-z_1)\vec{k}.$$

Kako $\overrightarrow{T_1T}$ leži u ravnini π, on je okomit na vektor normale $\vec{n}$ pa slijedi:

$$\begin{aligned}\vec{n}\cdot(\overrightarrow{T_1T}) &= 0\\ \left(A\vec{i}+B\vec{j}+C\vec{k}\right)\cdot\left((x-x_1)\vec{i}+(y-y_1)\vec{j}+(z-z_1)\vec{k}\right) &= 0\end{aligned}$$

pa je vektorska jednadžba ravnine zadane točkom $T_1=(x_1,y_1,z_1)$ i vektorom normale $\vec{n}=A\vec{i}+B\vec{j}+C\vec{k}$:

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0$$

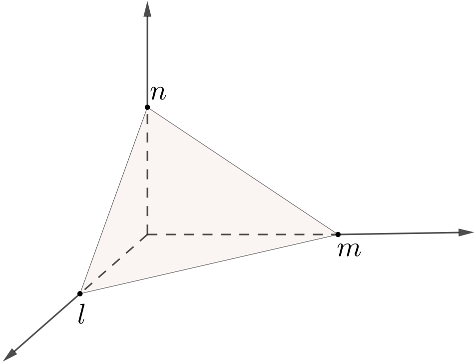
Kanonski oblik jednadžbe ravnine je:

$$Ax + By + Cz + D = 0$$

Ako je $D \neq 0$ dobivamo i segmentni oblik jednadžbe ravnine:

$$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 1,$$

gdje su l, m i n odsječci na koordinatnim osima.



Zadatak (2.7.)

Napišite jednadžbu ravnine koja prolazi:

a) ishodištem $O(0,0,0)$ i ima vektor normale $\vec{n} = (2,1,0)$ (sami).

b) točkom $M(1,0,-1)$ i ima vektor normale $\vec{n} = (0,1,2)$.

c) točkom $T(1,-2,0)$ i okomita je na dužinu $\overline{TS}$, gdje su $T(1,-2,0)$ i $S(0,-1,1)$.

Rješenje: a) Ravnina π određena je s $\vec{n} = (2,1,0)$ i točkom $O(0,0,0)$:

$$\begin{aligned} A(x-x_1)+B(y-y_1)+C(z-z_1) &= 0 \\ 2(x-0)+1(y-0)+0(z-0) &= 0 \\ \pi \equiv 2x+y &= 0 \end{aligned}$$

b) Ravnina π određena je s $\vec{n} = (0,1,2)$ i točkom $M(1,0,-1)$:

$$\begin{aligned} A(x-x_1)+B(y-y_1)+C(z-z_1) &= 0 \\ 0(x-1)+1(y-0)+2(z+1) &= 0 \\ \pi \equiv y+2z+2 &= 0 \end{aligned}$$

c) $\overline{TS} \perp \pi$ pa je $\vec{n} = \overrightarrow{TS} = -\vec{i} + \vec{j} + \vec{k}$. Ravnina π određena je s $\vec{n} = (-1, 1, 1)$ i točkom $T(1, -2, 0)$

$$\begin{aligned} A(x - x_1) + B(y - y_1) + C(z - z_1) &= 0 \\ -1(x - 1) + 1(y + 2) + 1(z - 0) &= 0 \\ \pi \equiv -x + y + z + 3 &= 0 \end{aligned}$$

Neka su $\pi_1 = (T_1, \vec{n_1})$ i $\pi_2 = (T_2, \vec{n_2})$, gdje su $\vec{n_1} = A_1\vec{i} + B_1\vec{j} + C_1\vec{k}$ i $\vec{n_2} = A_2\vec{i} + B_2\vec{j} + C_2\vec{k}$.

Ravnine su okomite ako su okomiti njihovi vektori normale, tj.

$$\vec{n_1} \perp \vec{n_2} \Longrightarrow \pi_1 \perp \pi_2$$

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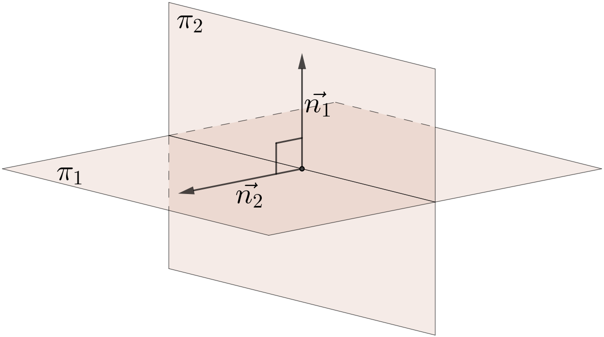

Ravnine su usporedne ako su njihovi vektori normale kolinearni, tj.

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$$

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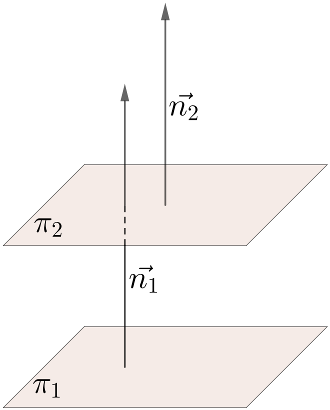


Zadatak (2.8.)

Odredite jednadžbu ravnine koja prolazi

a) točkom $T(1,0,2)$ i okomita je na os x.
b) točkom $T(2,1,2)$ i usporedna je s $xy-$ravninom.
c) točkom $T(2,-1,3)$ i usporedna je s osima y i z.
d) točkom $T(1,3,1)$ i osi x.
e) točkama $T(3,0,2)$, $S(1,2,-1)$ i usporedna je s osi x.

Rješenje:

a) Ravnina π određena je s $T(1,0,2)$ i vektorom normale $\vec{n} = \vec{i}$

$1(x-1) = 0$, tj. $\pi \equiv x - 1 = 0$

b) Jer je π usporedna s $xy-$ravninom, a $\vec{i}, \vec{j} \perp \vec{i} \times \vec{j}$, njezin je vektor normale $\vec{n} = \vec{i} \times \vec{j} = \vec{k}$.

Ravnina π određena je točkom $T(2,1,2)$ i normalom $\vec{n} = \vec{k}$

$\pi \equiv z - 2 = 0$

c) Jer je π usporedna s osima y i z, a $\vec{j}, \vec{k} \perp \vec{j} \times \vec{k}$, njezin je vektor normale $\vec{n} = \vec{j} \times \vec{k} = \vec{i}$.

Ravnina π određena je točkom $T(2, -1, 3)$ i normalom $\vec{n} = \vec{i}$

$\pi \equiv x - 2 = 0$

d) Jer ravnina π sadrži os x, sadrži i točku $O(0, 0, 0)$, pa sadrži i vektor $\overrightarrow{OT} = \vec{i} + 3\vec{j} + \vec{k}$. $\vec{i}, \overrightarrow{OT} \subset \pi$ i $\vec{i}, \overrightarrow{OT} \perp \overrightarrow{i} \times \overrightarrow{OT}$, dobivamo da je

$$
\begin{aligned}
\vec{n} &= \overrightarrow{i} \times \overrightarrow{OT} \\
&= \vec{i} \times \left(\vec{i} + 3\vec{j} + \vec{k}\right) \\
&= \underbrace{\vec{i} \times \vec{i}}_{\vec{0}} + 3\vec{i} \times \vec{j} + \vec{i} \times \vec{k} \\
&= 3\vec{k} - \vec{j} \\
&= -\vec{j} + 3\vec{k}
\end{aligned}
$$

$\pi \equiv -y + 3z = 0$

e) Jer ravnina π sadrži vektor $\overrightarrow{ST} = 2\vec{i} - 2\vec{j} + 3\vec{k}$ i usporedna je s osi x, dobivamo da je

$$\begin{aligned}
\vec{n} &= \vec{i} \times \overrightarrow{ST} \\
&= \vec{i} \times \left(2\vec{i} - 2\vec{j} + 3\vec{k}\right) \\
&= \underbrace{2\vec{i} \times \vec{i}}_{\vec{0}} - 2\vec{i} \times \vec{j} + 3\vec{i} \times \vec{k} \\
&= -2\vec{k} - 3\vec{j} \\
&= -3\vec{j} - 2\vec{k}
\end{aligned}$$

$-3(y-2) - 2(z+1) = 0$
$-3y + 6 - 2z - 2 = 0$
$\pi \equiv 3y + 2z - 4 = 0$

Zadatak (2.9.)

Odredite jednadžbu ravnine π koja je

a) okomita na ravnine $\pi_1 \equiv 2x - y + z - 2 = 0$ i $\pi_2 \equiv x + z + 1 = 0$ i prolazi točkom $T(1, 2, -1)$ (sami).

b) okomita na ravninu $\pi_1 \equiv 3x - 2y + z - 3 = 0$ i prolazi točkama $T(2, 1, 3)$ i $S(1, 0, -1)$.

Rješenje:

a) $\pi \perp \pi_1, \pi_2 \Longrightarrow \vec{n} \perp \vec{n_1}, \vec{n_2}$ i $\vec{n_1} = (2, -1, 1)$, $\vec{n_2} = (1, 0, 1)$ pa je:

$$\begin{aligned}
\vec{n} &= \vec{n_1} \times \vec{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 1 & 0 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} \\
&= \vec{i} \cdot (-1 - 0) - \vec{j} \cdot (2 - 1) + \vec{k} \cdot (0 + 1) \\
&= -\vec{i} - \vec{j} + \vec{k} \Longrightarrow \vec{n} = (-1, -1, 1)
\end{aligned}$$

$$-1(x-1)-1(y-2)+(z+1)=0$$
$$-x+1-y+2+z+1=0$$
$$\pi \equiv -x-y+z+4=0$$

b) $\pi \perp \pi_1$ i $\overrightarrow{TS} \subset \pi \Longrightarrow \vec{n} \perp \vec{n_1}, \overrightarrow{TS}$ i $\vec{n_1} = (3, -2, 1)$, $\overrightarrow{TS} = (-1, -1, -4)$ pa imamo:

$$\begin{aligned}
\vec{n} &= \vec{n_1} \times \vec{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ -1 & -1 & -4 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -2 & 1 \\ -1 & -4 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 3 & 1 \\ -1 & -4 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (8 + 1) - \vec{j} \cdot (-12 + 1) + \vec{k} \cdot (-3 - 2) \\
&= 9\vec{i} + 11\vec{j} - 5\vec{k} \Longrightarrow \vec{n} = (9, 11, -5)
\end{aligned}$$

$9(x - 1) + 11(y - 0) - 5(z + 1) = 0$
$9x - 9 + 11y - 5z - 5 = 0$
$\pi \equiv 9x + 11y - 5z - 14 = 0$

Zadatak (2.10.)

Odredite jednadžbu ravnine određenu točkama $M(1,-1,2)$, $N(3,2,0)$ i $P(1,-2,1)$.

Rješenje:

$M, N, P \in \pi$ pa slijedi da je $\overrightarrow{MN}, \overrightarrow{MP} \subset \pi$, odnosno $\vec{n} \perp \overrightarrow{MN}, \overrightarrow{MP}$

$\overrightarrow{MN} = 2\vec{i} + 3\vec{j} - 2\vec{k}$, $\overrightarrow{MP} = -\vec{j} - \vec{k}$

$$\begin{aligned}
\vec{n} &= \overrightarrow{MN} \times \overrightarrow{MP} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -2 \\ 0 & -1 & -1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 3 & -2 \\ -1 & -1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 2 & -2 \\ 0 & -1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (-3-2) - \vec{j} \cdot (-2-0) + \vec{k} \cdot (-2-0) \\
&= -5\vec{i} + 2\vec{j} - 2\vec{k} \Longrightarrow \vec{n} = (-5, 2, -2)
\end{aligned}$$

$$-5(x-1)+2(y+1)-2(z-2)=0$$
$$-5x+5+2y+2-2z+4=0$$
$$\pi \equiv -5x+2y-2z+11=0$$

Neka je $M = (x_0, y_0, z_0)$ točka i $\pi \equiv Ax + By + Cz + D = 0$ ravnina.
Udaljenost točke M od ravnine π je:

$$d(M, \pi) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

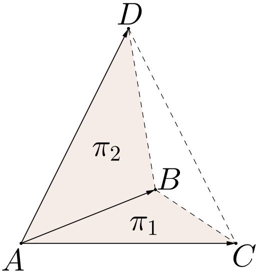
Kut među ravninama $\pi_1 = (T_1, \vec{n_1})$ i $\pi_2 = (T_2, \vec{n_2})$ je kut što ga zatvaraju njihove normale $\vec{n_1} = (A_1, B_1, C_1)$ i $\vec{n_2} = (A_2, B_2, C_2)$, tj.
$\varphi = \angle(\pi_1, \pi_2) = \angle(\vec{n_1}, \vec{n_2}), \varphi \in \left[0, \frac{\pi}{2}\right]$.

$$\cos\varphi = \frac{\vec{n_1} \cdot \vec{n_2}}{|\vec{n_1}| \cdot |\vec{n_2}|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Zadatak (2.11.)

Točke $A(2,5,0)$, $B(1,6,2)$, $C(-1,4,1)$ i $D(1,4,3)$ određuju tetraedar. Odredite kut među stranama ABC i ABD.

Rješenje:



$$\begin{aligned} \varphi &= \angle(\pi_1(A,B,C), \pi_2(A,B,D)) \\ &= \angle(\vec{n_1}, \vec{n_2}) \end{aligned}$$

$$
\begin{aligned}
\vec{n_1} &= \overrightarrow{AB} \times \overrightarrow{AC} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -3 & -1 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (1+2) - \vec{j} \cdot (-1+6) + \vec{k} \cdot (1+3) \\
&= 3\vec{i} - 5\vec{j} + 4\vec{k} \Longrightarrow \vec{n_1} = (3, -5, 4)
\end{aligned}
$$

$$
\begin{aligned}
\vec{n_2} &= \overrightarrow{AB} \times \overrightarrow{AD} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ -1 & -1 & 3 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} -1 & 2 \\ -1 & 3 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (3+2) - \vec{j} \cdot (-3+2) + \vec{k} \cdot (1+1) \\
&= 5\vec{i} + \vec{j} + 2\vec{k} \Longrightarrow \vec{n_2} = (5, 1, 2)
\end{aligned}
$$

$$
\begin{aligned}
\cos\varphi &= \frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|}\\
&= \frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{A_1^2+B_1^2+C_1^2}\cdot\sqrt{A_2^2+B_2^2+C_2^2}}\\
&= \frac{3\cdot 5-5\cdot 1+4\cdot 2}{\sqrt{3^2+5^2+4^2}\cdot\sqrt{5^2+1^2+2^2}}\\
&= \frac{15-5+8}{\sqrt{50}\cdot\sqrt{30}}\\
&= \frac{18}{10\sqrt{15}}\\
\varphi &= 62^\circ 17'
\end{aligned}
$$

Zadatak (2.12.)

Odredite jednadžbu ravnine π koja sadrži sve točke jednako udaljene od:

a) dviju ravnina $\pi_1 \equiv x - 3y - 2z + 1 = 0$ i $\pi_2 \equiv 2x - y + 3z + 3 = 0$.

b) dviju točaka $T_1(2, -1, 3)$ i $T_2(1, 2, -1)$.

Rješenje: a) Za svaku $T(x, y, z) \in \pi$ vrijedi $d(T, \pi_1) = d(T, \pi_2)$.

$$\frac{|A_1 x + B_1 y + C_1 z + D_1|}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = \frac{|A_2 x + B_2 y + C_2 z + D_2|}{\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

$$\frac{|x - 3y - 2z + 1|}{\sqrt{1^2 + 3^2 + 2^2}} = \frac{|2x - y + 3z + 3|}{\sqrt{2^2 + 1^2 + 3^2}} / \cdot \sqrt{14}$$

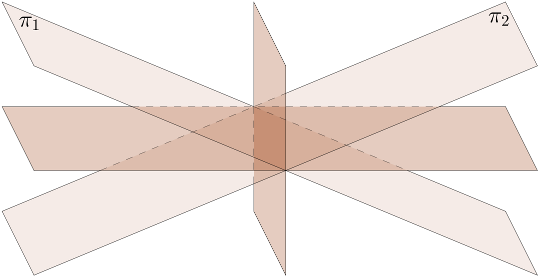
$$|x - 3y - 2z + 1| = |2x - y + 3z + 3|$$

1. slučaj:

$x - 3y - 2z + 1 = 2x - y + 3z + 3 \Longrightarrow \pi \equiv -x - 2y - 5z - 2 = 0$

2. slučaj:

$x - 3y - 2z + 1 = -2x + y - 3z - 3 \Longrightarrow \pi \equiv 3x - 4y + z + 4 = 0$


π_1
π_2

b) Za svaku točku $T(x, y, z) \in \pi$ vrijedi $d(T, T_1) = d(T, T_2)$.

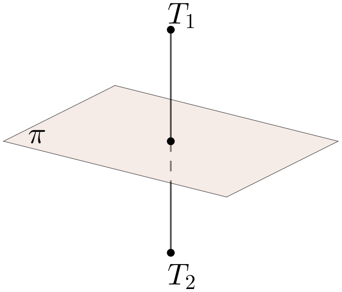
$$\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2} = \sqrt{(x-x_2)^2+(y-y_2)^2+(z-z_2)^2}$$

$$\sqrt{(x-2)^2+(y+1)^2+(z-3)^2} = \sqrt{(x-1)^2+(y-2)^2+(z+1)^2} \ /^2$$

$$-4x+4+2y+1-6z+9 = -2x+1-4y+4+2z+1$$

$$-2x+6y-8z+8 = 0/:(-2)$$

$$\pi \equiv x-3y+4z-4 = 0$$



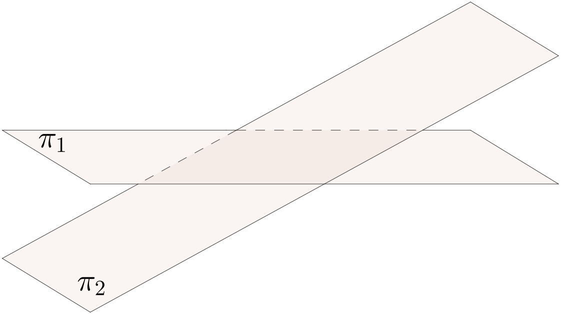
Zadatak (2.13.)

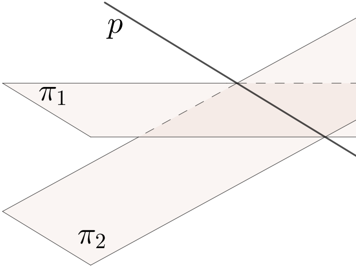
Pravac p zadan je kao presječnica dviju ravnina

$$p \equiv \begin{cases} x + y - 4z - 5 = 0 \\ 2x - y - 2z - 1 = 0 \end{cases}$$

Odredite njegovu jednadžbu u kanonskom i parametarskom obliku.

Rješenje:





Kako je $p(T, \vec{c}) \subset \pi_1, p \subset \pi_2$, imamo da je $\vec{n_1} = (1, 1, -4) \perp \vec{c}$ i $\vec{n_2} = (2, -1, -2) \perp \vec{c}$, odnosno $\vec{c} = \vec{n_1} \times \vec{n_2}$.

$$
\begin{aligned}
\vec{c} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -4 \\ 2 & -1 & -2 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & -4 \\ -1 & -2 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (-2 - 4) - \vec{j} \cdot (-2 + 8) + \vec{k} \cdot (-1 - 2) \\
&= -6\vec{i} - 6\vec{j} - 3\vec{k}
\end{aligned}
$$

Možemo pisati i $\vec{c} = -\frac{1}{3}\left(-6\vec{i} - 6\vec{j} - 3\vec{k}\right) = 2\vec{i} + 2\vec{j} + \vec{k}$, jer su kolinearni.

Odredimo sada neku točku $T \in \pi_1 \cap \pi_2$, npr. za $z = -1$

$$\begin{array}{rcr} x + y - 4z - 5 & = & 0 \\ 2x - y - 2z - 1 & = & 0 \\ \hline x + y + 4 - 5 & = & 0 \\ 2x - y + 2 - 1 & = & 0 \\ \hline x + y & = & 1 \\ 2x - y & = & -1 \\ \hline \end{array}$$

$$3x = 0 \Longrightarrow x = 0 \Longrightarrow y = 1$$

$$T(0, 1, -1) \in \pi_1 \cap \pi_2$$

$$p(T, \vec{c}) \equiv \frac{x}{2} = \frac{y-1}{2} = \frac{z+1}{1} \quad \text{i} \quad p \equiv \begin{cases} x = 2t \\ y = 1 + 2t \\ z = -1 + t \end{cases}$$

Zadatak (2.14.)

Odredite jednadžbu ravnine π koja

a) sadrži točku $T(1,1,1)$ i okomita je na pravac
$$p \equiv \frac{x-1}{2} = \frac{y}{-1} = \frac{z-3}{1}$$

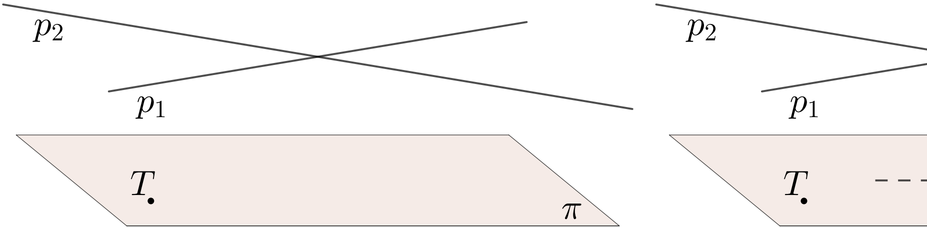
b) sadrži točku $T(1,1,1)$ i usporedna je s pravcima
$$p_1 \equiv \frac{x-2}{1} = \frac{y+1}{0} = \frac{z-1}{1} \text{ i } p_2 \equiv \frac{x+1}{2} = \frac{y}{1} = \frac{z+1}{-1} \text{ (sami).}$$

Rješenje:

a) Kako je $\pi \perp p$, imamo da je $\vec{n} = \vec{c} = (2,-1,1)$

$$2(x-1) - 1(y-1) + 1(z-1) = 0$$
$$\pi \equiv 2x - y + z - 2 = 0$$

b)



Kako su $p_1(T_1, \vec{c_1}), p_2(T_2, \vec{c_2}) || \pi$ imamo da je $\vec{n} \perp \vec{c_1}, \vec{c_2}$.

$$
\begin{aligned}
\vec{n} &= \overrightarrow{c_1} \times \overrightarrow{c_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\
&= \vec{i} \cdot (0 - 1) - \vec{j} \cdot (-1 - 2) + \vec{k} \cdot (1 - 0) \\
&= -\vec{i} + 3\vec{j} + \vec{k} \Longrightarrow \vec{n} = (-1, 3, 1)
\end{aligned}
$$

$$-1(x-1)+3(y-1)+1(z-1)=0$$
$$\pi \equiv -x+3y+z-3=0$$

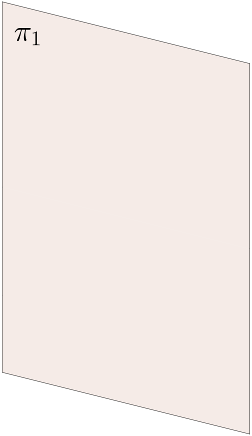
Zadatak (2.15.)

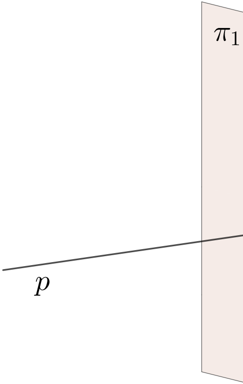
Odredite jednadžbu ravnine π koja sadrži pravac

$$p \equiv \frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{1}$$

i okomita je na ravninu $\pi_1 \equiv 2x + 3y + z + 1 = 0$.

Rješenje:





Označimo sa $\vec{n}$ normalu ravnine π, a s $\vec{m} = (2,3,1)$ normalu ravnine π_1. Kako je $p \subset \pi$, imamo da je $\vec{c} = (1,1,1) \perp \vec{n}$, a iz $\pi \perp \pi_1$ imamo da je $\vec{n} \perp \vec{m}$

$$
\begin{aligned}
\vec{n} &= \overrightarrow{c} \times \overrightarrow{m} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \\
&= \vec{i} \cdot (1-3) - \vec{j} \cdot (1-2) + \vec{k} \cdot (3-2) \\
&= -2\vec{i} + \vec{j} + \vec{k} \Longrightarrow \vec{n} = (-2,1,1)
\end{aligned}
$$

$$-2(x-1) + (y-1) + 1(z-1) = 0$$

$$\pi \equiv -2x + y + z = 0$$

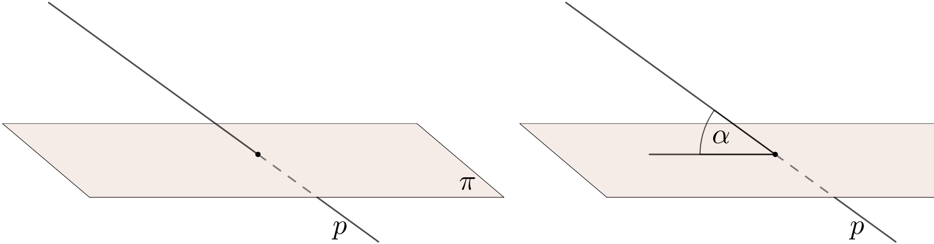
Zadatak (2.16.)

Odredite kut što ga zatvaraju pravac

$$p \equiv \begin{cases} 3x + y - z + 1 = 0 \\ 2x - y + 4z - 2 = 0 \end{cases}$$

i ravnina $\pi \equiv x - 8y + 3z - 6 = 0$.

Rješenje:



$\vec{c} \perp \vec{n_1} = (3, 1, -1), \vec{n_2} = (2, -1, 4)$

$$
\begin{aligned}
\vec{c} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & -1 \\ 2 & -1 & 4 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & -1 \\ -1 & 4 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 3 & -1 \\ 2 & 4 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} \\
&= \vec{i} \cdot (4 - 1) - \vec{j} \cdot (12 + 2) + \vec{k} \cdot (-3 - 2) \\
&= 3\vec{i} - 14\vec{j} - 5\vec{k} = (3, -14, -5)
\end{aligned}
$$

$$\vec{n} = \vec{i} - 8\vec{j} + 3\vec{k} = (1, -8, 3)$$

$$\varphi = \angle(\vec{n}, \vec{c})$$

$$\begin{aligned}
\cos\varphi &= \frac{\vec{n}\cdot\vec{c}}{|\vec{n}|\cdot|\vec{c}|} \\
&= \frac{1\cdot 3 - 8\cdot 14 + 3\cdot(-5)}{\sqrt{1^2+(-8)^2+3^2}\cdot\sqrt{3^2+14^2+(-5)^2}} \\
&= \frac{3+112-15}{\sqrt{74}\cdot\sqrt{230}} \\
&= \frac{100}{\sqrt{74\cdot 230}} \\
\varphi &= 39^\circ 57'
\end{aligned}$$

$$\begin{aligned}
\alpha &= \angle(\pi, p) \\
&= 90^\circ - \varphi \\
&= 90^\circ - 39^\circ 57' \\
&= 50^\circ 3'
\end{aligned}$$

Zadatak (2.17.)

Odredite presjek pravca p i ravnine π ako je:

a) $p \equiv \dfrac{x-2}{1} = \dfrac{y}{-2} = \dfrac{z+1}{-1}$ i $\pi \equiv 2x - 3y + z + 4 = 0$

b) $p \equiv \dfrac{x-2}{1} = \dfrac{y}{-2} = \dfrac{z+1}{-1}$ i $\pi \equiv 2x - 3y + 8z + 1 = 0$

c) $p \equiv \dfrac{x-2}{1} = \dfrac{y}{-2} = \dfrac{z+1}{-1}$ i $\pi \equiv 2x - 3y + 8z + 4 = 0$

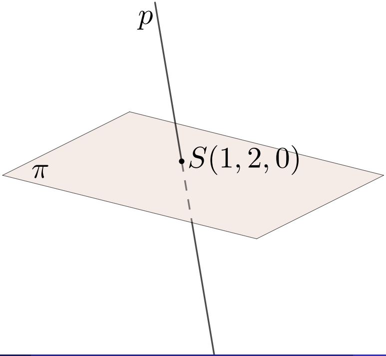
Rješenje:

a) Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku:$p \equiv \begin{cases} x = 2 + t \\ y = -2t \\ z = -1 - t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv 2x - 3y + z + 4 = 0$

$$\begin{aligned}
2x - 3y + z + 4 &= 0\\
2(2+t) - 3(-2t) + (-1-t) + 4 &= 0\\
4 + 2t + 6t - 1 - t + 4 &= 0\\
7t &= -7\\
t &= -1
\end{aligned}$$

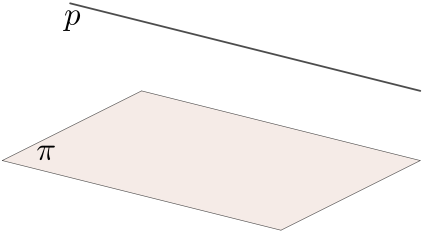
$$p \equiv \begin{cases} x = 2 + (-1) = 1 \\ y = -2(-1) = 2 \\ z = -1 - (-1) = 0 \end{cases} \Longrightarrow p \cap \pi = S(1,2,0)$$



b) Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku: $p \equiv \begin{cases} x = 2 + t \\ y = -2t \\ z = -1 - t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv 2x - 3y + 8z + 1 = 0$

$$\left.\begin{aligned} 2x - 3y + 8z + 1 &= 0 \\ 2(2+t) - 3(-2t) + 8(-1-t) + 1 &= 0 \\ 4 + 2t + 6t - 8 - 8t + 1 &= 0 \\ -3 &\neq 0 \end{aligned}\right. \implies \pi || p$$

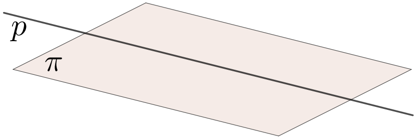


c) Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku: $p \equiv \begin{cases} x = 2 + t \\ y = -2t \\ z = -1 - t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv 2x - 3y + 8z + 4 = 0$

$$\begin{aligned} 2x - 3y + 8z + 4 &= 0 \\ 2(2+t) - 3(-2t) + 8(-1-t) + 4 &= 0 \\ 4 + 2t + 6t - 8 - 8t + 4 &= 0 \\ 0 &= 0, \forall t \in \mathbb{R} \end{aligned}$$

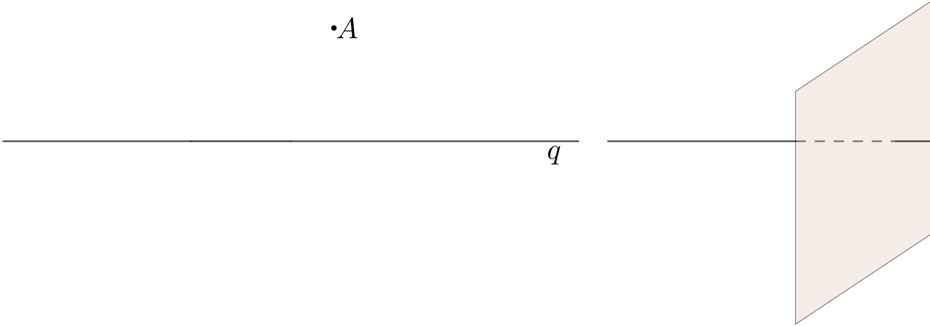
$\Longrightarrow p \cap \pi = p$...pravac leži u ravnini



Zadatak (2.18.)

Odredite jednadžbu pravca p koji prolazi točkom $A(2,-3,1)$, okomit je na pravac $q \equiv \frac{x-3}{2} = \frac{y+3}{-1} = \frac{z-5}{3}$ i siječe ga.

Rješenje:



Odredimo jednadžbu ravnine π koja je okomita na q i sadrži točku A, tj. $A \in \pi$ i $\pi \perp q$.

Tada je $\vec{n} = \vec{c_q} = (2, -1, 3)$, pa je $2(x-2) - (y+3) + 3(z-1) = 0$

$\pi \equiv 2x - y + 3z - 10 = 0$

Odredimo sada točku $S = \pi \cap q$. Parametarski oblik jednadžbe pravca

$$q \equiv \begin{cases} x = 3 + 2t \\ y = -3 - t \\ z = 5 + 3t \end{cases}$$

Uvrstimo je u jednadžbu ravnine $\pi \equiv 2x - y + 3z - 10 = 0$

$$\begin{aligned} 2x - y + 3z - 10 &= 0 \\ 2(3+2t) - (-3-t) + 3(5+3t) - 10 &= 0 \\ 6 + 4t + 3 + t + 15 + 9t - 10 &= 0 \\ 14t + 14 &= 0 \\ t &= -1 \end{aligned}$$

$$q \equiv \begin{cases} x = 3 + 2(-1) = 1 \\ y = -3 - (-1) = -2 \\ z = 5 + 3(-1) = 2 \end{cases}$$

$\pi \cap q = S(1, -2, 2)$

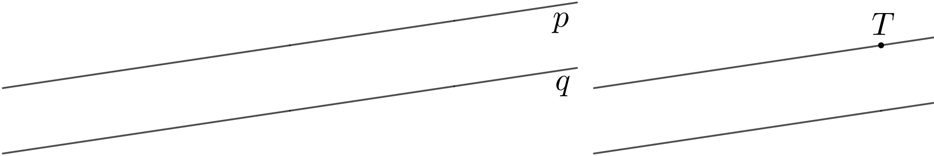
Odredimo sada jednadžbu pravca p koji prolazi točkama A i S.

$\vec{c} = \overrightarrow{AS} = (-1, 1, 1)$, pa je $p \equiv \dfrac{x-2}{-1} = \dfrac{y+3}{1} = \dfrac{z-1}{1}$.

Zadatak (2.19.)

Odredite udaljenost između dva usporedna pravca
$p \equiv \frac{x-1}{2} = \frac{y+2}{3} = \frac{z}{-1}$ i $q \equiv \frac{x-1}{2} = \frac{y-1}{3} = \frac{z+5}{-1}$.

Rješenje:



Odredimo po volji jednu točku $T \in p$, npr. $T(1, -2, 0)$. Odredimo sada jednadžbu ravnine π, za koju vrijedi da je $T \in \pi$ i $\pi \perp p$. Tada je $d(p, q) = d(T, S)$, gdje je $S = \pi \cap q$.

Kako je $\pi \perp p$ imamo da je $\vec{n} = \vec{c} = (2, 3, -1)$

$$2(x-1) + 3(y+2) - 1(z-0) = 0$$
$$\pi \equiv 2x + 3y - z + 4 = 0$$

Odredimo sada točku $S = \pi \cap q$. Parametarski oblik jednadžbe pravca

$$q \equiv \begin{cases} x = 1 + 2t \\ y = 1 + 3t \\ z = -5 - t \end{cases}$$

Uvrstimo je u jednadžbu ravnine $\pi \equiv 2x + 3y - z + 4 = 0$

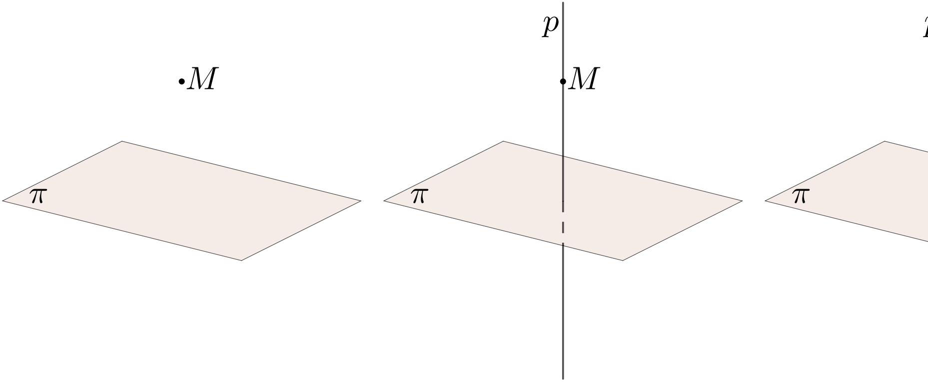
$$\begin{aligned} 2x + 3y - z + 4 &= 0 \\ 2(1 + 2t) + 3(1 + 3t) - (-5 - t) + 4 &= 0 \\ 2 + 4t + 3 + 9t + 5 + t + 4 &= 0 \\ t &= -1 \end{aligned}$$

$$\Longrightarrow q \equiv \begin{cases} x = 1 + 2(-1) = -1 \\ y = 1 + 3(-1) = -2 \\ z = -5 - (-1) = -4 \end{cases} \Longrightarrow \pi \cap q = S(-1, -2, -4)$$

$$
\begin{aligned}
d(p,q) &= d(T,S)\\
&= \sqrt{(x_S - x_T)^2 + (y_S - y_T)^2 + (z_S - z_T)^2}\\
&= \sqrt{(-1-1)^2 + (-2+2)^2 + (-4-0)^2}\\
&= \sqrt{4+0+16}\\
&= \sqrt{20}\\
&= 2\sqrt{5}
\end{aligned}
$$

Zadatak (2.20.)

Odredite koordinate točke N koja je simetrična točki $M(1,1,1)$ obzirom na ravninu $\pi \equiv x + y - 2z - 6 = 0$.



Odredimo koordinate točke M', projekcije točke M na ravninu π.
Odredimo i jednadžbu pravca p za kojeg vrijedi da je $M \in p$ i $p \perp \pi$. Kako je $p \perp \pi$ imamo da je $\vec{c} = \vec{n} = (1, 1, -2)$ pa je $p \equiv \begin{cases} x = 1 + t \\ y = 1 + t \\ z = 1 - 2t \end{cases}$

$M' = p \cap \pi$

$$\begin{aligned} x + y - 2z - 6 &= 0 \\ 1 + t + 1 + t - 2(1 - 2t) - 6 &= 0 \\ 2 + 2t - 2 + 4t - 6 &= 0 \\ 6t &= 6 \\ t &= 1 \end{aligned} \implies p \equiv \begin{cases} x = 1 + 1 = 2 \\ y = 1 + 1 = 2 \\ z = 1 - 2 = -1 \end{cases}$$

$\pi \cap p = M'(2, 2, -1)$

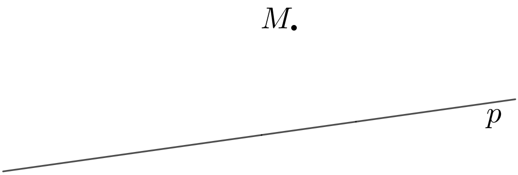
Kako je M' polovište dužine $\overline{MN}$ imamo da je
$M' = \left(\frac{x_M + x_N}{2}, \frac{y_M + y_N}{2}, \frac{z_M + z_N}{2} \right)$

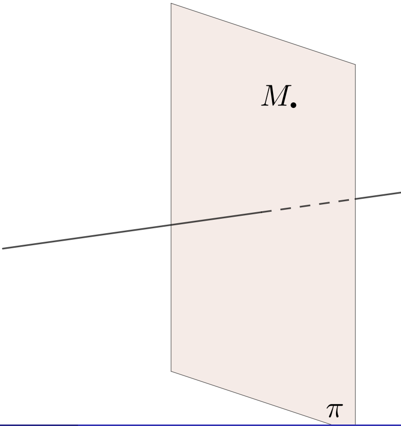
$$\left.\begin{aligned} 2 &= \frac{1 + x_N}{2} \Longrightarrow x_N = 3 \\ 2 &= \frac{1 + y_N}{2} \Longrightarrow y_N = 3 \\ -1 &= \frac{1 + z_N}{2} \Longrightarrow z_N = -3 \end{aligned}\right\} \Longrightarrow N(3, 3, -3)$$

Zadatak (2.21.)

Odredite točku N koja je simetrična točki $M(1,0,2)$ obzirom na pravac $p \equiv \dfrac{x-2}{3} = \dfrac{y}{5} = \dfrac{z+1}{1}$.

Rješenje:





Odredimo jednadžbu ravnine π za koju vrijedi da je $\pi \perp p$ i $M \in \pi$. Kako je $\pi \perp p$ imamo da je $\vec{n} = \vec{c} = (3, 5, 1)$.

$$3(x-1) + 5(y-0) + 1(z-2) = 0$$
$$\pi \equiv 3x + 5y + z - 5 = 0$$

Odredimo sada koordinate točke $M' = p \cap \pi$.

Parametarski oblik jednadžbe pravca $p \equiv \begin{cases} x = 2 + 3t \\ y = 5t \\ z = -1 + t \end{cases}$

Uvrstimo je u jednadžbu ravnine $\pi \equiv 3x + 5y + z - 5 = 0$

$$\begin{aligned} 3x + 5y + z - 5 &= 0 \\ 3(2+3t) + 5(5t) + (-1+t) - 5 &= 0 \\ 6 + 9t + 25t - 1 + t - 5 &= 0 \\ 35t &= 0 \\ t &= 0 \end{aligned}$$

$$p \equiv \begin{cases} x = 2 + 3 \cdot 0 = 2 \\ y = 5 \cdot 0 = 0 \\ z = -1 + 0 = -1 \end{cases} \implies p \cap \pi = M'(2, 0, -1)$$

Kako je M' polovište dužine $\overline{MN}$ imamo da je

$$M' = \left(\frac{x_M + x_N}{2}, \frac{y_M + y_N}{2}, \frac{z_M + z_N}{2} \right)$$

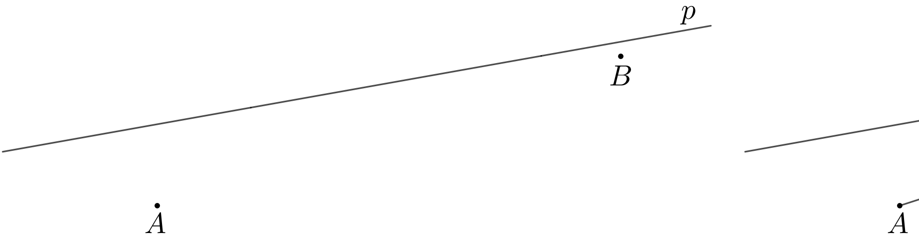
$$\left. \begin{aligned} 2 &= \frac{1 + x_N}{2} \implies x_N = 3 \\ 0 &= \frac{0 + y_N}{2} \implies y_N = 0 \\ -1 &= \frac{2 + z_N}{2} \implies z_N = -4 \end{aligned} \right\} \implies N(3, 0, -4)$$

Zadatak (2.22.)

Na pravcu $p \equiv x = y = z$ odredite točku T koja je jednako udaljena od točaka $A(2,4,0)$ i $B(4,0,4)$.

Rješenje: Odredimo jednadžbu simetralne ravnine dužine $\overline{AB}$, tj. $\pi_S(P, \vec{n})$, gdje je P polovište dužine $\overline{AB}$, a $\vec{n} = \overrightarrow{AB} = (2,-4,4)(\pi_S \perp \overline{AB})$.

$$P = \left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}, \frac{z_A + z_B}{2}\right) = (3,2,2)$$



$2(x-3)-4(y-2)+4(z-2)=0$
$2x-4y+4z-6=0/:2$
$\pi_S \equiv x-2y+2z-3=0$

Parametarski oblik jednadžbe pravca p je $p \equiv \begin{cases} x=t \\ y=t \\ z=t \end{cases}$

Odredimo sada koordinate točke $T = p \cap \pi_S$

$$\begin{aligned} x-2y+2z-3 &= 0 \\ t-2t+2t-3 &= 0 \\ t &= 3 \end{aligned} \implies p \equiv \begin{cases} x=3 \\ y=3 \\ z=3 \end{cases}$$

$\pi_S \cap p = T(3,3,3)$

Zadatak (2.23.)

Odredite udaljenost točke $A(4,-5,4)$ od ravnine određene pravcima zadanima kao presjek ravnina

$$p_1 \equiv \begin{cases} \pi_1 ... x - y + z - 3 = 0 \\ \pi_2 ... x + y + z - 1 = 0 \end{cases} \text{ i } p_2 \equiv \begin{cases} \pi_3 ... y = 0 \\ \pi_4 ... x + z = 0 \end{cases}$$

.

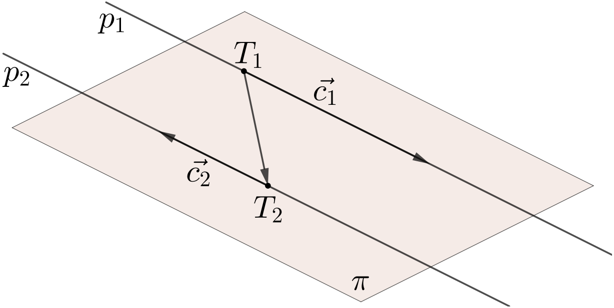
Rješenje: Odredimo vektore smjera pravaca p_1 i p_2.

$$\begin{aligned}
\vec{c_1} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\
&= \vec{i} \cdot (-1-1) - \vec{j} \cdot (1-1) + \vec{k} \cdot (1+1) \\
&= -2\vec{i} + 2\vec{k} = (-2, 0, 2)
\end{aligned}$$

$$
\begin{aligned}
\vec{c_2} &= \overrightarrow{n_3} \times \overrightarrow{n_4} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\
&= \vec{i} \cdot (1 - 0) - \vec{j} \cdot (0 - 0) + \vec{k} \cdot (0 - 1) \\
&= \vec{i} - \vec{k} = (1, 0, -1)
\end{aligned}
$$

Uočimo da je $\vec{c_1} = -2\vec{c_2}$ pa izlazi da su kolinearni, tj. $\vec{c_1} \times \vec{c_2} = \vec{0}$

$$\left(\vec{n} = \overrightarrow{c_1} \times \overrightarrow{c_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 2 \\ 1 & 0 & -1 \end{vmatrix} = \vec{i} \cdot (0-0) - \vec{j} \cdot (2-2) + \vec{k} \cdot (0-0) = \vec{0} \right)$$



Odredimo točku $T_1 \in p_1$, npr. za $z = 0$ i točku $T_2 \in p_2$, npr. za $z = -1$

$$\begin{array}{rcl} x - y + z - 3 & = & 0 \\ x + y + z - 1 & = & 0 \\ \hline x - y & = & 3 \\ x + y & = & 1 \\ \hline 2x & = & 4 \\ x = 2 & \text{i} & y = -1 \\ T_1(2, -1, 0) & & \end{array} \qquad \begin{array}{rcl} y & = & 0 \\ x + z & = & 0 \\ \hline y & = & 0 \\ x - 1 & = & 0 \\ \hline y & = & 0 \\ x & = & 1 \\ T_2(1, 0, -1) & & \end{array}$$

Sada je normala ravnine π jednaka $\vec{n} = \vec{c_2} \times \overrightarrow{T_1 T_2}$

$$\begin{aligned} \vec{n} &= \vec{c_2} \times \overrightarrow{T_1 T_2} \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ -1 & 1 & -1 \end{vmatrix} \\ &= \vec{i} \cdot (0 + 1) - \vec{j} \cdot (-1 - 1) + \vec{k} \cdot (0 + 1) \\ &= \vec{i} + 2\vec{j} + \vec{k} = (1, 2, 1) \end{aligned}$$

Jednadžba ravnine π je:

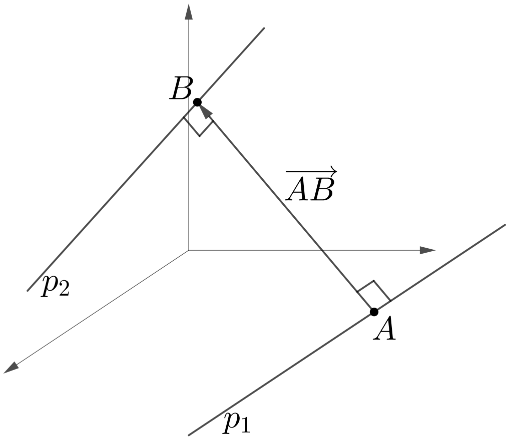
$1(x-2)+2(y+1)+z=0$
$\pi \equiv x+2y+z=0$

$$
\begin{aligned}
d(A,\pi) &= \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}} \\
&= \frac{|1\cdot 4-5\cdot 2+4\cdot 1|}{\sqrt{1^2+2^2+1^2}} \\
&= \frac{|4-10+4|}{\sqrt{1+4+1}} \\
&= \frac{2}{\sqrt{6}} \\
&= \frac{2\sqrt{6}}{6} \\
&= \frac{\sqrt{6}}{3}
\end{aligned}
$$

Zadatak (2.24.)

Na pravcu $p_1 \equiv \dfrac{x-6}{3} = \dfrac{y+2}{-1} = \dfrac{z-4}{-2}$ odredite točku A koja je najbliža pravcu $p_2 \equiv \begin{cases} x = t \\ y = t \\ z = 2 \end{cases}$.

Rješenje: Među svim vektorima $\overrightarrow{AB}$, gdje je $A \in p_1$, a $B \in p_2$ moramo pronaći onaj za kojeg vrijedi da je $\overrightarrow{AB} \perp p_1$ i $\overrightarrow{AB} \perp p_2$. Tada će $d(A, B)$ biti najmanja.



B
$\overrightarrow{AB}$
p_2
A
p_1

Kako je $p_1 \equiv \begin{cases} x = 3s + 6 \\ y = -s - 2 \\ z = -2s + 4 \end{cases}$, tada je $A(3s + 6, -s - 2, -2s + 4)$, za neki $s \in \mathbb{R}$, proizvoljna točka pravca p_1.

Točka $B(t, t, 2)$, za neki $t \in \mathbb{R}$, je proizvoljna točka pravca p_2.

$\overrightarrow{AB} = (t - 3s - 6)\vec{i} + (t + s + 2)\vec{j} + (2 + 2s - 4)\vec{k}$.

Trebamo pronaći takve t i s da vrijedi: $\overrightarrow{AB} \perp p_1$ i $\overrightarrow{AB} \perp p_2$. No, onda imamo da je $\overrightarrow{AB} \perp \vec{c_1} = (3, -1, -2)$ i $\overrightarrow{AB} \perp \vec{c_2} = (1, 1, 0)$, odnosno $\overrightarrow{AB} \cdot \vec{c_1} = 0$ i $\overrightarrow{AB} \cdot \vec{c_2} = 0$

$$
\begin{array}{rcl}
\overrightarrow{AB}\cdot\vec{c_1} & = & 0\\
\overrightarrow{AB}\cdot\vec{c_2} & = & 0\\
\hline
3(t-3s-6)-(t+s+2)-2(2+2s-4) & = & 0\\
t-3s-6+t+s+2 & = & 0\\
\hline
3t-9s-18-t-s-2+4-4s & = & 0\\
2t-2s & = & 4\\
\hline
t-7s & = & 8\\
t-s & = & 2\\
\hline
t-7s-(t-s) & = & 8-2\\
-6s & = & 6\\
s & = & -1\\
t & = & 1
\end{array}
$$

$A(3s+6,-s-2,-2s+4) \Longrightarrow A\,(3,-1,6)$

$B(t,t,2) \Longrightarrow B\,(1,1,2).$

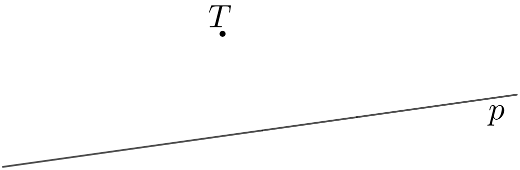
Zadatak (2.25.)

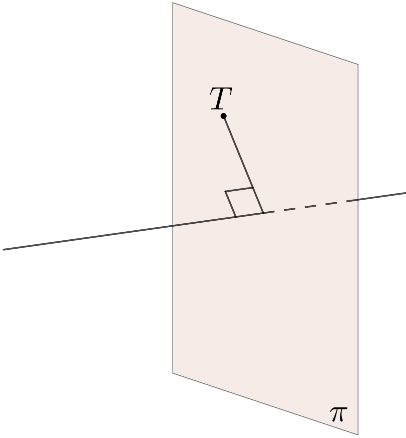
Odredite ortgonalnu projekciju točke $T(0, 5, -1)$ na pravac

$$p \equiv \begin{cases} x + y = 0 \\ x - z + 3 = 0 \end{cases}.$$

Rješenje: Odredimo vektor smjera $\vec{c}$ pravca p.

$$\begin{aligned}
\vec{c} &= \overrightarrow{n_1} \times \overrightarrow{n_2} \\
&= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \\
&= \vec{i} \cdot \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - \vec{j} \cdot \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + \vec{k} \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \\
&= \vec{i} \cdot (-1 - 0) - \vec{j} \cdot (-1 - 0) + \vec{k} \cdot (0 - 1) \\
&= -\vec{i} + \vec{j} - \vec{k} = (-1, 1, -1)
\end{aligned}$$


T
p


T
π

Za proizvoljnu točku $T_1 \in p$, npr. za $x = 0$ imamo $T_1(0, 0, 3)$ i parametarski oblik jednadžbe pravca $p \equiv \begin{cases} x = -t \\ y = t \\ z = 3 - t \end{cases}$

Odredimo sada jednadžbu ravnine π za koju vrijedi $\pi \perp p$ i $T \in \pi$. Tada je $\vec{n} = \vec{c} = (-1, 1, -1)$.

$$-1(x-0)+1(y-5)-1(z+1)=0$$
$$\pi \equiv -x+y-z-6=0$$

Sada je ortogonalna projekcija točke T, točka $P=\pi \cap p$.

Odredimo sada koordinate točke $P=p \cap \pi$

$$\begin{aligned} -x+y-z-6 &= 0 \\ -(-t)+t-(3-t)-6 &= 0 \\ t+t-3+t-6 &= 0 \\ 3t &= 9 \\ t &= 3 \end{aligned} \Longrightarrow p \equiv \begin{cases} x=-3 \\ y=3 \\ z=0 \end{cases}$$

$$\pi \cap p = P(-3,3,0)$$

Zadatak (2.26.)

Odredite $A \in \mathbb{R}$ za koji je ravnina $\pi \equiv Ax + 3y - 5z - 31 = 0$ usporedna s pravcem $p \equiv \dfrac{x-1}{4} = \dfrac{y+1}{3} = z$. Odredite projekciju pravca p na ravninu π.

Rješenje:

$\pi \equiv Ax + 3y - 5z - 31 = 0$ pa je $\vec{n} = A\vec{i} + 3\vec{j} - 5\vec{k} = (A, 3, -5)$.

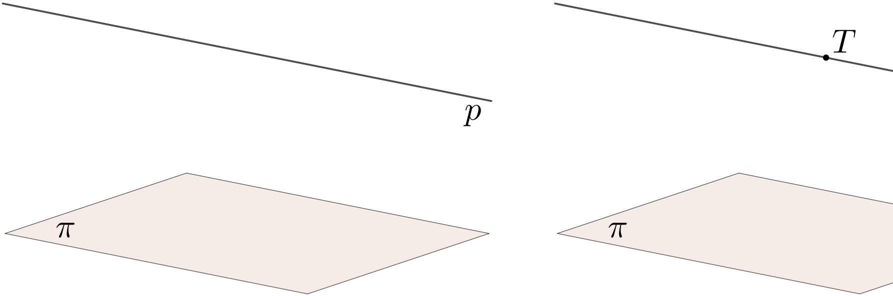
$p \equiv \dfrac{x-1}{4} = \dfrac{y+1}{3} = \dfrac{z}{1}$ pa je $\vec{c} = 4\vec{i} + 3\vec{j} + 1\vec{k} = (4, 3, 1)$

$\pi || p \Longrightarrow \vec{n} \perp \vec{c} \Longleftarrow \vec{n} \cdot \vec{c} = 0$

$$\begin{aligned}
\vec{n} \cdot \vec{c} &= 0 \\
4A + 3 \cdot 3 - 5 \cdot 1 &= 0 \\
4A + 9 - 5 &= 0 \\
4A &= -4 \\
A &= -1 \Longrightarrow \vec{n} = (-1, 3, -5)
\end{aligned}$$

$\pi \equiv -x + 3y - 5z - 31 = 0$

Odaberimo proizvoljno neku točku $T \in p$, npr. za $x = 1$ imamo $T(1, -1, 0)$. Odredimo jednadžbu pravca q za kojeg vrijedi $q \perp \pi$ i $T \in q$.



$$\vec{c_q} || \vec{n} \Longrightarrow \vec{c_q} = \vec{n} = (-1, 3, -5), \text{ pa je } q \equiv \begin{cases} x = 1 - t \\ y = -1 + 3t \\ z = -5t \end{cases}$$

Odredimo koordinate točke $\overline{T} = \pi \cap q$.

$$\begin{aligned} -x + 3y - 5z - 31 &= 0 \\ -(1-t) + 3(-1+3t) - 5(-5t) - 31 &= 0 \\ -1 + t - 3 + 9t + 25t - 31 &= 0 \implies q \equiv \\ 35t - 35 &= 0 \\ t &= 1 \end{aligned}$$

$$\begin{cases} x = 1 - 1 = 0 \\ y = -1 + 3 = 2 \\ z = -5 \end{cases}$$

$\pi \cap p = \overline{T}(0, 2, -5)$

Ortogonalna projekcija pravca p je $\overline{p} = (\overline{T}, \vec{c})$

$$\overline{p} \equiv \frac{x}{4} = \frac{y-2}{3} = \frac{z+5}{1}$$

Zadatak (2.27)

Odredite međusobni položaj pravca $p \equiv \dfrac{x-2}{-2} = \dfrac{y-3}{-3} = \dfrac{z-6}{-2}$ i ravnine $\pi \equiv x + 2y - z + 4 = 0$. Odredite otrogonalnu projekciju pravca p na ravninu π.

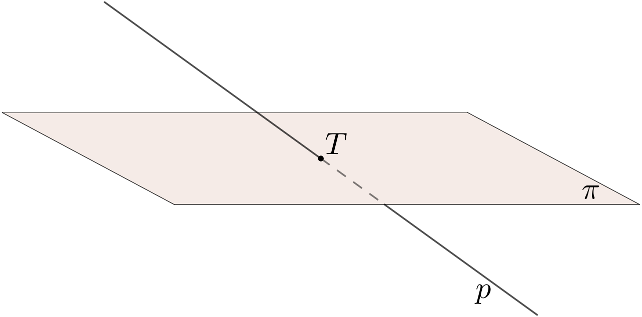
Odredimo $p \cap \pi$. Jednadžba pravca p u parametarskom obliku:

$$p \equiv \begin{cases} x = 2 - 2t \\ y = 3 - 3t \\ z = 6 - 2t \end{cases}$$

Uvrstimo je u jednadžnu ravnine $\pi \equiv x + 2y - z + 4 = 0$

$$\begin{aligned} x + 2y - z + 4 = 0 &= 0 \\ 2 - 2t + 2(3 - 3t) - (6 - 2t) + 4 &= 0 \\ 2 - 2t + 6 - 6t - 6 + 2t + 4 &= 0 \\ -6t &= -6 \\ t &= 1 \end{aligned}$$

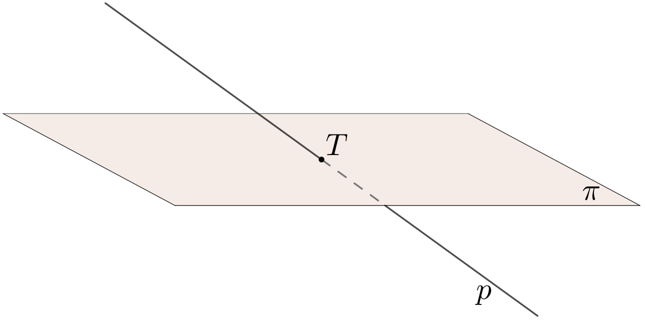
$$p \equiv \begin{cases} x = 2 - 2 \cdot 1 = 0 \\ y = 3 - 3 \cdot 1 = 0 \\ z = 6 - 2 \cdot 1 = 4 \end{cases}$$



Pravac p probada ravninu π u točki $T(0, 0, 4)$.

Točka probodišta T je sama sebi ortogonalna projekcija. Dovoljno je odabrati još samo jednu točku, npr. za $t = 0$, imamo točku $S(2, 3, 6)$. Odredimo i jednadžbu pravca q koji je okomit na ravninu π i prolazi točkom S. Dakle, $q \perp \pi$ pa je $\vec{n} = \vec{c_q} = (1, 2, -1)$.

Odredimo koordinate probodišta pravca q i ravnine π. To će biti ortogonalna projekcija točke S na ravninu π.



Jednadžba pravca q u parametarskom obliku: $q \equiv \begin{cases} x = 2 + t \\ y = 3 + 2t \\ z = 6 - t \end{cases}$

Uvrstimo je u jednadžnu ravnine $\pi \equiv x + 2y - z + 4 = 0$

$$\begin{aligned} x + 2y - z + 4 = 0 &= 0 \\ 2 + t + 2(3 + 2t) - (6 - t) + 4 &= 0 \\ 2 + t + 6 + 4t - 6 + t + 4 &= 0 \\ 6t &= -6 \\ t &= -1 \end{aligned}$$

$$q \equiv \begin{cases} x = 2 + (-1) = 1 \\ y = 3 + 2 \cdot (-1) = 1 \\ z = 6 - (-1) = 7 \end{cases}$$

Pravac q probada ravninu π u točki $S'(1, 1, 7)$.

Jednadžbu pravca p' dobivamo kroz dvije točke, $T(0,0,4)$ i $S'(1,1,7)$:

$$p' \equiv \frac{x}{1} = \frac{y}{1} = \frac{z-4}{3}$$